



An Unstable Two-Phase Membrane Problem and Maximum Flux Exchange Flow

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Abstract Let U be a bounded open connected set in \mathbb{R}^n ($n \geq 1$). We refer to the unique weak solution of the Poisson problem $-\Delta u = \chi_A$ on U with Dirichlet boundary conditions as u_A for any measurable set A in U . The function $\psi := u_U$ is the torsion function of U . Let V be the measure $V := \psi \mathcal{L}^n$ on U where \mathcal{L}^n stands for n -dimensional Lebesgue measure. We study the variational problem

$$I(U, p) := \sup \left\{ J(A) - V(U) p^2 \right\}$$

with $p \in (0, 1)$ where $J(A) := \int_A u_A dx$ and the supremum is taken over measurable sets $A \subset U$ subject to the constraint $V(A) = pV(U)$. We relate the above problem to an unstable two-phase membrane problem. We characterise optimisers in the case $n = 1$. The proof makes use of weighted isoperimetric and Pólya–Szegő inequalities.

Keywords Two-phase membrane problem · Isoperimetric inequality · Pólya–Szegő inequality · Spherical cap symmetrisation

Mathematics Subject Classification 26D10 · 35J20 · 35J60

1 Introduction and Motivation

Let U be a bounded open connected set in \mathbb{R}^n ($n \geq 1$). We refer to the unique weak solution of the Poisson problem

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$$-\Delta u = \chi_A \text{ on } U, \quad u \in W_0^{1,2}(U), \quad (1.1)$$

as u_A for any measurable set A in U . The function $\psi := u_U$ is the torsion function of U . Let V be the measure $V := \psi \mathcal{L}^n$ on U where \mathcal{L}^n stands for the n -dimensional Lebesgue measure. For $p \in (0, 1)$ consider the variational problem

$$I(U, p) := \sup \left\{ J(A) - V(U) p^2 \right\} \quad (1.2)$$

where $J(A) := (u_A, \chi_A)$ and the supremum is taken over measurable sets $A \subset U$ subject to the constraint $V(A) = pV(U)$. Here, (\cdot, \cdot) stands for the usual inner product in the real Hilbert space $L^2(U)$. Any maximiser E in (1.2) will be called an *optimal configuration* for the data (U, p) . If E is an optimal configuration and $u = u_E$, then (u, E) will be called an *optimal pair*.

In Corollary 2.2 we show that for each $p \in (0, 1)$ the problem (1.2) admits an optimal pair (u, E) for the data (U, p) . In Proposition 3.3 we characterise the optimal configuration E as a super level set of u/ψ ; that is, $E = \{u > c\psi\}$ for some $c \in (0, 1)$ up to \mathcal{L}^n -a.e. equivalence. The derivation assumes that U is a $C^{1,1}$ domain. Under this last assumption, we show in Corollary 3.4 that u satisfies the following semi-linear elliptic partial differential equation with discontinuous nonlinearity. Put $v := u - c\psi$ with c as above. Then v is a strong solution of the problem

$$-\Delta v = (1 - c)\chi_{\Omega_+(v)} - c\chi_{\Omega_-(v)} \text{ on } U$$

where $\Omega_{\pm}(v) := \{\pm v > 0\}$. The above equation is similar to Problem C (the two-phase membrane problem) in [20, 1.2.3] but with a sign change; see also the unstable membrane problem [20] 2.5. It is noted in [20, 1.1.7] that the composite membrane problem (see [6, 7]) is akin to the unstable membrane problem. Our terminology is adopted from [6, 7] and in places there is a similarity in method. The regular part of the free boundary $\Gamma(v) = \partial\Omega_{\pm}(v) \cap U$ is real-analytic (Theorem 3.7). In Sect. 4 we replace U with the unit ball B in \mathbb{R}^n ($n \geq 2$). For $p \in (0, 1)$ we show that any optimal configuration E for the data (B, p) possesses spherical cap symmetry \mathcal{L}^n -a.e. (see Theorem 4.1).

In the remainder of the article, we study the problem (1.2) in the one-dimensional case $n = 1$ and take $B = (-1, 1)$. In Theorem 9.5 we show that any optimal configuration E with data (B, p) is \mathcal{L}^1 -a.e. equivalent to an open interval abutting a boundary point of B . A first step in obtaining this result is to transform the problem using an analog of the ground-state transformation (with the torsion function in place of the ground-state) (see Proposition 9.2). We then obtain an isoperimetric inequality on B with volume density ψ and perimeter density $\psi^{3/2}$ (Theorem 6.3) and a corresponding Hardy-Littlewood type inequality (Theorem 6.6) and a Pólya-Szegő inequality (Theorem 7.10). We also study the case of equality in the isoperimetric and Pólya-Szegő inequalities (Theorem 6.4 and Corollary 8.7 respectively). We have been guided by [2] in obtaining these results, though our setting and proofs are slightly different.

We have not obtained an analog of Theorem 9.5 in the case $n \geq 2$. At least part of our method transfers to higher dimensions. There is a counterpart of the isoperimetric

inequality Theorem 6.3 (though its derivation is more involved with the usual difficulties around regularity and stability) and the Hardy–Littlewood inequality is a ready consequence. A potential stumbling block is the validity of a corresponding Pólya–Szegő inequality. We note that the sufficient conditions given in [22] are stringent. The problem (1.2) is related to maximum flux exchange flow (a model of magma flow in a volcanic vent [15]). We take $n = 2$ and consider a configuration of two immiscible fluids in a vertical duct with cross-section U in a state of steady flow. The densities of the fluids are labelled ρ, ρ' with $\rho > \rho'$ and each fluid has unit viscosity. The pressure p has constant gradient $\partial p / \partial z = -G$ on U . Suppose the fluid with density ρ occupies a region A in U . By the Navier–Stokes equations, the vertical component of velocity u satisfies

$$\begin{aligned} 0 &= \Delta u + G - \rho g \quad \text{on } A; \\ 0 &= \Delta u + G - \rho' g \quad \text{on } U \setminus A. \end{aligned}$$

Dirichlet boundary conditions are imposed on the boundary of U and it is assumed that u and its gradient are continuous on the interface between the two regions A and $U \setminus A$.

The parameter G lies in the interval $(\rho'g, \rho g)$ which allows the possibility of bidirectional flow. On rescaling (and relabelling the velocities) we obtain the system

$$\begin{aligned} 0 &= \Delta u - \lambda - 1 \quad \text{on } A; \\ 0 &= \Delta u - \lambda + 1 \quad \text{on } U \setminus A; \end{aligned} \tag{1.3}$$

where

$$\lambda := \frac{(\rho' + \rho)g - 2G}{(\rho - \rho')g} \in (-1, 1)$$

is a proxy for the pressure gradient. Two problems arise: one to maximise the flux $(\chi_{U \setminus A}, u)$ over regions A which satisfy the flux balance condition $(u, 1) = 0$ with constant λ ; the other in which we optimize also over λ . In detail,

$$\gamma(U) := \sup \{ (\chi_{U \setminus A}, u) : (u, 1) = 0, A \subset U \text{ open}, \lambda \in (-1, 1) \}, \tag{1.4}$$

$$\gamma(U, \lambda) := \sup \{ (\chi_{U \setminus A}, u) : (u, 1) = 0, A \subset U \text{ open} \}, \tag{1.5}$$

where in the latter λ is fixed in the interval $(-1, 1)$. In the case $n = 1$ and $U = B$ we show that any optimal configuration E for the problem (1.5) with data (B, λ) is \mathcal{L}^1 -a.e. equivalent to an open interval abutting a boundary point of B in Theorem 9.8. Moreover, any optimal configuration E for the problem (1.4) is \mathcal{L}^1 -a.e. equivalent to either $(-1, 0)$ or $(0, 1)$.

2 Existence of Optimal Configurations

Define

$$\mathcal{V}_t := \left\{ f \in L^2(U) : 0 \leq f \leq 1 \text{ } \mathcal{L}^n\text{-a.e. on } U \text{ and } (f, \psi) \leq t \right\}$$

for $t \in (0, V(U))$ and consider the variational problem

$$\beta(U, t) := \sup \left\{ J(f) : f \in \mathcal{V}_t \right\} \quad (2.1)$$

where $J(f) := (u_f, f)$ and u_f is the unique solution of the Poisson problem (1.1) but with inhomogeneity f . The first main result runs as follows.

Theorem 2.1 *Fix $t \in (0, V(U))$. Then*

- (i) *there exists $f \in \mathcal{V}_t$ such that $\beta(U, t) = J(f)$;*
- (ii) *$(\psi, f) = t$;*
- (iii) *f has the form $f = \chi_E$ for some measurable set E in U .*

Corollary 2.2 *For each $p \in (0, 1)$ the problem (1.2) admits an optimal pair (u, E) for the data (U, p) .*

Proof Let $p \in (0, 1)$ and put $t := pV(U)$. Let E be as in Theorem 2.1 (iii). Then $V(E) = (\psi, \chi_E) = t = pV(U)$. Let $A \subset U$ be a measurable set with $V(A) = pV(U)$. Then $f = \chi_A \in \mathcal{V}_t$ so $J(E) \geq J(A)$. \square

We prepare a few lemmas before proving Theorem 2.1.

Lemma 2.3 *Let X, Y be (real) Banach spaces and suppose that $X \subset Y$ with continuous embedding. Let (x_h) be a sequence in X which converges weakly in X to $x \in X$. Then (x_h) converges weakly to x in Y .*

Proof Note that for any $g \in Y'$, $g|_X \in X'$. \square

We remark that the Dirichlet Laplacian $(D(\Delta), \Delta)$ is associated with the Dirichlet form $(\mathcal{F}, \mathcal{E})$ in $L^2(U)$ with form domain $\mathcal{F} := W_0^{1,2}(U)$ and

$$\mathcal{E}(u, v) = \int_U \nabla u \cdot \nabla v \, dx \quad (u, v \in \mathcal{F}).$$

Let G stand for the corresponding Green operator.

Lemma 2.4 *Let $t \in (0, V(U))$. Then*

- (i) *the functional $J : \mathcal{V}_t \rightarrow \mathbb{R}$ is continuous in the topology of weak sequential convergence;*
- (ii) *$J : \mathcal{V}_t \rightarrow \mathbb{R}$ is convex.*

Proof (i) Suppose that a sequence (f_h) in \mathcal{V}_t converges weakly to an element $f \in \mathcal{V}_t$ in $L^2(U)$. Put $u_h := Gf_h \in L^2(U)$. For each h and $\varphi \in L^2(U)$,

$$(u_h, \varphi) = (Gf_h, \varphi) = (f_h, G\varphi),$$

by symmetry of G so that $(u_h, \varphi) \rightarrow (f, G\varphi) = (u, \varphi)$ as $h \rightarrow \infty$ where $u := Gf$. We also have that

$$\mathcal{E}(u_h, \varphi) = (f_h, \varphi) \rightarrow (f, \varphi) = \mathcal{E}(u, \varphi) \text{ as } h \rightarrow \infty$$

for any $\varphi \in \mathcal{F}$. This means that (u_h) converges weakly to u in \mathcal{F} .

Note that $W_0^{1,2}(U) \subset W_0^{1,1}(U)$ and $\|u\|_{W_0^{1,1}(U)} \leq \sqrt{2|U|} \|u\|_{W_0^{1,2}(U)}$ for each $u \in W_0^{1,2}(U)$. By Lemma 2.3, (u_h) converges weakly to u in $W_0^{1,1}(U)$.

If $n \geq 2$ we may use the Rellich–Kondrachov compactness theorem [10, 5.7] for example and [16, Theorem 21.2.9] to conclude that (u_h) converges strongly to u in $L^1(U)$. Now

$$J(f) - J(f_h) = (u, f) - (u_h, f_h) = (u, f - f_h) + (f_h, u - u_h)$$

and the right-hand side converges to zero as $h \rightarrow \infty$ as (f_h) is bounded in $L^\infty(U)$. This shows that $J(f_h) \rightarrow J(f)$ as $h \rightarrow \infty$.

In the case $n = 1$ we use the fact that $W_0^{1,2}(U)$ is compactly embedded in $C^0(\overline{U})$ (see [13, Theorem 7.22]) and hence in $L^1(U)$.

(ii) Let $f \in \mathcal{V}_t$. By Dirichlet's principle [10, 2.2.5] for example,

$$E(f) := \inf_{v \in \mathcal{F}} \left\{ (1/2) \mathcal{E}(v, v) - (v, f) \right\} = -(1/2) J(f).$$

The functional E is concave so that J is convex. \square

Lemma 2.5 Let $t \in (0, V(U))$. A function f in the convex set $\mathcal{V}_t \subset L^2(U)$ is extremal only if $f = \chi_A$ \mathcal{L}^n -a.e. on U for some $A \subset U$ measurable with $(\psi, \chi_A) \leq t$.

Proof The proof runs as in [11, Lemma 2]. A measurable function f on U is \mathcal{L}^n -a.e. equivalent to χ_A for some $A \subset U$ measurable if and only if $f(1-f) = 0$ \mathcal{L}^n -a.e. on U . Suppose that $f \in \mathcal{V}_t$ is an extremal element and assume that $|\{f(1-f) \neq 0\}| > 0$ for a contradiction. Then there exists $\varepsilon > 0$ and a measurable set E in U with positive \mathcal{L}^n -measure such that $\varepsilon \leq f \leq 1 - \varepsilon$ on E . Decompose E into two disjoint sets E_1, E_2 each with positive \mathcal{L}^n -measure. Choose $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \setminus \{0\}$ such that $\alpha_1(\psi, \chi_{E_1}) + \alpha_2(\psi, \chi_{E_2}) = 0$ and define

$$f_\tau := f + \tau \sum_{j=1}^2 \alpha_j \chi_{E_j}$$

for $\tau \in \mathbb{R}$. Then $f_\tau \in \mathcal{V}_t$ for $|\tau| \leq \varepsilon/|\alpha_1| \vee |\alpha_2|$. We then derive the contradiction that f is not extremal as $f = (1/2) \{f_\tau + f_{-\tau}\}$ for such τ . \square

Proof of Theorem 2.1 Let (f_h) be a maximising sequence for $\beta(U, t)$. Now \mathcal{V}_t is weakly sequentially compact in $L^2(U)$. This follows by appeal to [16, Theorem 10.2.9] due to the fact that \mathcal{V}_t is bounded, closed and convex in the reflexive Banach space $L^2(U)$. So we may assume that (f_h) converges weakly in $L^2(U)$ to some $f \in \mathcal{V}_t$ as $h \rightarrow \infty$ after choosing a subsequence if necessary. By Lemma 2.4 (i),

$$\beta(U, t) = \lim_{h \rightarrow \infty} J(f_h) = J(f),$$

giving item (i) of the Theorem. It is straightforward to see that $(\psi, f) = t$ and hence (ii).

We now argue as in [8, Corollary 6.2]. By [5, Chapitre II §7 Proposition 1.1 (EVT II.58)], J attains its supremum on \mathcal{V}_t at an extremal point f . We then invoke Lemma 2.5 to conclude that f has the form $f = \chi_A \mathcal{L}^n$ -a.e. on U for some measurable set A in U and hence (iii). \square

3 Some Partial Regularity Results

Proposition 3.1 *Suppose that U is a $C^{1,1}$ domain. Let $E \subset U$ be a measurable set and $u = u_E$. Then $V(\{u = t\psi\}) = 0$ for each $t \in (0, 1)$.*

Proof By [13, Theorem 9.15], $u \in W^{2,p}(U)$ for any $1 < p < \infty$. Put $v := u - t\psi$, $N_t := \{v = 0\}$ and assume that $|N_t| > 0$. By [13, Lemma 7.7], we derive that $D^\alpha v = 0$ \mathcal{L}^n -a.e. and hence V -a.e. on N_t for any multi-index α with $|\alpha| = 1$. Observe that $D^\alpha v$ belongs to $W^{1,p}(U)$ for $|\alpha| \leq 1$. Applying the last-mentioned lemma once more, we see $D^\alpha v = 0$ V -a.e. on N_t for any multi-index α with $|\alpha| \leq 2$. So $-\Delta v = 0$ V -a.e. on N_t . But $-\Delta v = \chi_E - t \chi_U$ V -a.e. on U . This leads to a contradiction. \square

We require a version of the bathtub principle (see [17, Theorem 1.14]). Let (X, \mathcal{A}, μ) be a finite measure space and ρ a positive \mathcal{A} -measurable integrable function on X . Given $0 < v < \mu(X)$, consider the variational problem

$$\sup \int_X \chi_E \rho d\mu \quad (3.1)$$

where the supremum is taken over measurable sets $E \subset X$ with $\mu(E) = v$. We say that measurable sets A, B in X are equivalent μ -a.e. and write $A = B$ if and only if $\mu(A \Delta B) = 0$.

Theorem 3.2 *Assume that*

$$\mu(\{\rho = t\}) = 0 \text{ for all } t > 0. \quad (3.2)$$

Then for each $v \in (0, \mu(X))$ the problem (3.1) has a unique optimiser up to equivalence μ -a.e. given by $E = \{\rho > s\}$ where

$$s := \inf\{\tau > 0 : \mu(\{\rho > \tau\}) \leq v\}.$$

Proof The distribution function $\mu_\rho : (0, \infty) \rightarrow (0, V(U))$; $\tau \mapsto \mu(\{\rho > \tau\})$ is non-increasing and right-continuous on $(0, \infty)$; in fact, continuous thanks to (3.2). By right-continuity of μ_ρ , $\mu_\rho(s) \leq v$; by left-continuity, the reverse inequality holds, so $\mu(E) = \mu_\rho(s) = v$. For a measurable set A in X with $\mu(A) = v$,

$$\begin{aligned} \int_X \chi_A \rho \, d\mu &= \int_0^\infty \mu(A \cap \{\rho > \tau\}) \, d\tau \\ &= \int_0^s \mu(A \cap \{\rho > \tau\}) \, d\tau + \int_s^\infty \mu(A \cap \{\rho > \tau\}) \, d\tau \\ &\leq sv + \int_s^\infty \mu(\{\rho > \tau\}) \, d\tau = \int_X \chi_E \rho \, d\mu \end{aligned}$$

according to the layer cake representation [17, Theorem 1.13]. It follows that E is an optimiser for (3.1).

Suppose A is a measurable set in X with $\mu(A) = v$ which is not μ -a.e. equivalent to E . Then

$$\mu(E) = \mu(E \setminus A) + \mu(E \cap A) = v = \mu(A \setminus E) + \mu(E \cap A) = \mu(A)$$

so $\mu(E \setminus A) > 0$ as otherwise $\mu(A \setminus E) = 0$ and A is μ -a.e. equivalent to E . By countable additivity,

$$0 < \mu(E \setminus A) = \lim_{\tau \downarrow s} \mu(\{\rho > \tau\} \setminus A). \quad (3.3)$$

Thus,

$$\begin{aligned} \int_X \chi_A \rho \, d\mu &= \int_0^s \mu(A \cap \{\rho > \tau\}) \, d\tau + \int_s^\infty \mu(\{\rho > \tau\}) \, d\tau - \int_s^\infty \mu(\{\rho > \tau\} \setminus A) \, d\tau \\ &\leq sv + \int_s^\infty \mu(\{\rho > \tau\}) \, d\tau - \int_s^\infty \mu(\{\rho > \tau\} \setminus A) \, d\tau \\ &= \int_X \chi_E \rho \, d\mu - \int_s^\infty \mu(\{\rho > \tau\} \setminus A) \, d\tau < \int_X \chi_E \rho \, d\mu \end{aligned}$$

where the strict inequality follows from (3.3). \square

Let U be a $C^{1,1}$ domain and $p \in (0, 1)$. Let (u, E) be an optimal pair for (1.2) with data (U, p) . By [13, Corollary 9.18] we may assume that $u \in C^0(\overline{U})$.

Proposition 3.3 *Suppose that U is a $C^{1,1}$ domain. Let $p \in (0, 1)$ and suppose that (u, E) is an optimal pair for (1.2) with data (U, p) . Then $V(E \Delta \{u > c\psi\}) = V(E \Delta \{u \geq c\psi\}) = 0$ where $c \in (0, 1)$ is uniquely determined by the condition*

$$V(\{u > c\psi\}) = pV(U). \quad (3.4)$$

Proof Put $F := \{u > c\psi\}$ with c as in (3.4). Assume for a contradiction that $V(E \Delta F) > 0$. We consider a version of Problem (3.1) on U with ρ replaced by $w := u_E/\psi$ and μ replaced by V . By Proposition 3.1, $V(\{w = t\}) = 0$ for each $t > 0$; thus condition (3.2) holds. By uniqueness of the optimiser in Theorem 3.2 and the Cauchy–Schwarz inequality,

$$J(E) = (u_E, \chi_E) = \int_U w \chi_E dV < \int_U w \chi_F dV = \mathcal{E}(u_E, u_F) \leq J(E)^{1/2} J(F)^{1/2}$$

so that $J(E) < J(F)$, contradicting the assumption that E is an optimal configuration. The identity $V(E \Delta \{u \geq c\psi\}) = 0$ follows from Proposition 3.1. \square

Corollary 3.4 *Suppose that U is a $C^{1,1}$ domain. Let $p \in (0, 1)$ and suppose that (u, E) is an optimal pair for (1.2) with data (U, p) . Put $v := u - c\psi$ where c is given by (3.4). Then v is a strong solution of the problem*

$$-\Delta v = (1 - c)\chi_{\Omega_+(v)} - c\chi_{\Omega_-(v)} \text{ on } U \quad (3.5)$$

where $\Omega_{\pm}(v) := \{\pm v > 0\}$ and $E = \Omega_+(v)$ \mathcal{L}^n -a.e.

Proof By [13, Theorem 9.15], $u \in W^{2,p}(U)$ for any $1 < p < \infty$ and u is a strong solution of $-\Delta u = \chi_E$. By Proposition 3.3, u is a strong solution of $-\Delta u = \chi_{\{u > c\psi\}}$. The result follows from the fact that $-\Delta(c\psi) = c\chi_U$ and Proposition 3.1. \square

Lemma 3.5 *Let $p \in (0, 1)$ and (u, E) be an optimal pair for the data (U, p) . Then $(\psi - u, U \setminus E)$ is an optimal pair for the data $(U, 1 - p)$.*

Proof Let $A \subset U$ be a measurable set with $V(A) = pV(U)$. Then

$$J(U \setminus A) = J(A) - V(A) + V(U \setminus A) = J(A) + (1 - 2p)V(U)$$

so that $J(U \setminus A) - V(U)(1 - p)^2 = J(A) - V(U)p^2$ and the result follows. \square

Put $\Gamma_{\pm}(v) = \partial\Omega_{\pm}(v) \cap U$.

Lemma 3.6 *Suppose that U is a $C^{1,1}$ domain. Suppose that (u, E) is an optimal pair for the data (U, p) and let v be as in Corollary 3.4. Then $\Gamma_+(v) = \Gamma_-(v)$.*

Proof Suppose that $x \in \Gamma_+(v) \setminus \Gamma_-(v)$. Then there exists $r > 0$ such that $B(x, r) \subset U$, $u \geq c\psi$ on $B(x, r)$ and $u(x) = c\psi(x)$. By Proposition 3.3, $V(B(x, r) \setminus E) \leq V(\{u \geq c\psi\} \setminus E) = 0$ and $B(x, r) \setminus E$ is a Lebesgue null set. Let Φ stand for the fundamental solution of Laplace's equation in \mathbb{R}^n . By the mean-value formula (see [10, 2.5 Problem 3] for example), for any $0 < \tau < r$,

$$\begin{aligned}
u(x) &= |B(x, \tau)|^{-1} \int_{\partial B(x, \tau)} u \, d\mathcal{H}^{n-1} + \int_{B(x, \tau)} \left\{ \Phi(y - x) - \Phi(\tau\omega) \right\} \chi_E \, dy \\
&= |B(x, \tau)|^{-1} \int_{\partial B(x, \tau)} u \, d\mathcal{H}^{n-1} + \int_{B(x, \tau)} \left\{ \Phi(y - x) - \Phi(\tau\omega) \right\} \, dy \\
&\geq c |B(x, \tau)|^{-1} \int_{\partial B(x, \tau)} \psi \, d\mathcal{H}^{n-1} + \int_{B(x, \tau)} \left\{ \Phi(y - x) - \Phi(\tau\omega) \right\} \, dy \\
&> c \left\{ |B(x, \tau)|^{-1} \int_{\partial B(x, \tau)} \psi \, d\mathcal{H}^{n-1} + \int_{B(x, \tau)} \left\{ \Phi(y - x) - \Phi(\tau\omega) \right\} \, dy \right\} \\
&= c\psi(x)
\end{aligned}$$

as $c \in (0, 1)$, a contradiction. Here, ω is an arbitrary element in the unit sphere \mathbb{S}^{n-1} in \mathbb{R}^n .

Now suppose that $x \in \Gamma_-(v) \setminus \Gamma_+(v)$. As before, there exists $r > 0$ such that $u \leq c\psi$ on $B(x, r)$ and $u(x) = c\psi(x)$; alternatively, $\psi - u \geq (1 - c)\psi$ on $B(x, r)$ and $(\psi - u)(x) = (1 - c)\psi(x)$. By Lemma 3.5, $(\psi - u, U \setminus E)$ is an optimal pair for the data $(U, 1 - p)$. We then get a contradiction as above. \square

Put $\Gamma(v) := \Gamma_+(v) = \Gamma_-(v)$ and $\Gamma^*(v) := \Gamma(v) \cap \{|\nabla v| \neq 0\}$. The next theorem follows as in [20, Theorem 4.24].

Theorem 3.7 *Suppose that U is a $C^{1,1}$ domain. Suppose that (u, E) is an optimal pair for the data (U, p) and that $x_0 \in \Gamma^*(v)$. Then there exists $r > 0$ such that $\Gamma(v) \cap B(x, r)$ is a real-analytic hypersurface in $B(x, r)$.*

4 Spherical Cap Symmetry

In this section, we replace U by the open unit ball B in \mathbb{R}^n ($n \geq 2$) centred at the origin. We prove the following symmetry result. The notion of spherical cap symmetry is defined below.

Theorem 4.1 *Let $p \in (0, 1)$. Suppose that (u, E) is an optimal pair for the data (B, p) . Then E possesses spherical cap symmetry \mathcal{L}^n -a.e.*

We first discuss the operation of polarisation for integrable functions on B (see [4] and references therein). For $v \in \mathbb{S}^{n-1}$ the closed half-space $H = H_v$ is defined by

$$H_v := \{x \in \mathbb{R}^n : x \cdot v \geq 0\}$$

with an associated reflection $\tau_H : \mathbb{R}^n \rightarrow \mathbb{R}^n$; $x \mapsto x - 2(x \cdot v)v$. We refer to the collection of closed half-spaces H by \mathcal{H} . The polarisation f_H of $f \in L^1_+(B)$ with respect to $H \in \mathcal{H}$ is defined as follows. Choose an \mathcal{L}^n -version \tilde{f} of f . Set

$$\tilde{f}_H(x) := \begin{cases} \tilde{f}(x) \vee \tilde{f}(\tau_H x) & \text{for } x \in B \cap H, \\ \tilde{f}(x) \wedge \tilde{f}(\tau_H x) & \text{for } x \in B \setminus H; \end{cases}$$

\tilde{f}_H is \mathcal{L}^n -measurable and its \mathcal{L}^n -equivalence class $f_H := [\tilde{f}_H]$ is the polarisation of f . The definition is well-defined due to the fact that if $\tilde{f} = \tilde{g}$ \mathcal{L}^n -a.e. on B then $\tilde{f} = \tilde{g}$ \mathcal{H}^{n-1} -a.e. on \mathbb{S}_τ^{n-1} for \mathcal{L}^1 -a.e. $0 < \tau < 1$, and vice-versa. The Green kernel $G(x, y)$ for B is given by

$$G(x, y) = \Phi(y - x) - \Phi(|x|(y - x^*)) \text{ for } (x, y) \in B \times B \setminus \mathfrak{d},$$

where Φ is the fundamental solution of Laplace's equation in \mathbb{R}^n as before, \mathfrak{d} stands for the diagonal in $B \times B$ and the decoration $*$ refers to inversion in the unit sphere. We note the inequality

$$G(x, y) > G(\tau_H x, y) \text{ for any } x, y \in B \cap \text{int } H, \quad (4.1)$$

which follows from the strong maximum principle.

Theorem 4.2 *Let $f \in L^1_+(B)$ and $H \in \mathcal{H}$. Then $J(f) \leq J(f_H)$ with equality if and only if either $f = f_H$ or $f \circ \tau_H = f_H$ \mathcal{L}^n -a.e. on B .*

Proof Let \tilde{f} be an \mathcal{L}^n -version of f . Define

$$A^+ := \{x \in B \cap H : \tilde{f}(x) < \tilde{f}(\tau_H x)\}$$

and similarly S^+ but with the strict inequality replaced by the sign $>$. Put $A^- := \tau_H A^+$ and $A := A^+ \cup A^-$. In this notation,

$$\tilde{f}_H = \chi_A \tilde{f} \circ \tau_H + \chi_{B \setminus A} \tilde{f}.$$

As a consequence,

$$\begin{aligned} J(f_H) &= J(\chi_A \tilde{f} \circ \tau_H) + 2(\chi_A \tilde{f} \circ \tau_H, G\chi_{B \setminus A} \tilde{f}) + J(\chi_{B \setminus A} \tilde{f}) \\ &= J(\chi_A \tilde{f}) + 2(\chi_A \tilde{f} \circ \tau_H, G\chi_{B \setminus A} \tilde{f}) + J(\chi_{B \setminus A} \tilde{f}) \end{aligned}$$

and a similar identity holds for $J(f)$ but without composition with reflection. We may then write

$$\begin{aligned} J(f_H) - J(f) &= 2(\chi_A [\tilde{f} \circ \tau_H - \tilde{f}], G\chi_{B \setminus A} \tilde{f}) \\ &= 2 \int_{A^+} \int_{B \cap H \setminus A^+} (\tilde{f}(\tau_H x) - \tilde{f}(x)) G(x, y) \tilde{f}(y) dy dx \\ &\quad + 2 \int_{A^+} \int_{(B \setminus H) \setminus A^-} (\tilde{f}(\tau_H x) - \tilde{f}(x)) G(x, y) \tilde{f}(y) dy dx \\ &\quad + 2 \int_{A^-} \int_{B \cap H \setminus A^+} (\tilde{f}(\tau_H x) - \tilde{f}(x)) G(x, y) \tilde{f}(y) dy dx \\ &\quad + 2 \int_{A^-} \int_{(B \setminus H) \setminus A^-} (\tilde{f}(\tau_H x) - \tilde{f}(x)) G(x, y) \tilde{f}(y) dy dx \end{aligned}$$

$$\begin{aligned}
&= 2 \int_{A^+} \int_{B \cap H \setminus A^+} (\tilde{f}(\tau_H x) - \tilde{f}(x)) G(x, y) \tilde{f}(y) dy dx \\
&\quad + 2 \int_{A^+} \int_{B \cap H \setminus A^+} (\tilde{f}(\tau_H x) - \tilde{f}(x)) G(\tau_H x, y) \tilde{f}(\tau_H y) dy dx \\
&\quad - 2 \int_{A^+} \int_{B \cap H \setminus A^+} (\tilde{f}(\tau_H x) - \tilde{f}(x)) G(x, \tau_H y) \tilde{f}(y) dy dx \\
&\quad - 2 \int_{A^+} \int_{B \cap H \setminus A^+} (\tilde{f}(\tau_H x) - \tilde{f}(x)) G(x, y) \tilde{f}(\tau_H y) dy dx \\
&= 2 \int_{A^+} \int_{B \cap H \setminus A^+} (\tilde{f}(\tau_H x) - \tilde{f}(x)) (G(x, y) - G(\tau_H x, y)) \tilde{f}(y) dy dx \\
&\quad - 2 \int_{A^+} \int_{B \cap H \setminus A^+} (\tilde{f}(\tau_H x) - \tilde{f}(x)) (G(x, y) - G(\tau_H x, y)) \tilde{f}(\tau_H y) dy dx \\
&= 2 \int_{A^+} \int_{S^+} (\tilde{f}(\tau_H x) - \tilde{f}(x)) (G(x, y) - G(\tau_H x, y)) (\tilde{f}(y) - \tilde{f}(\tau_H y)) dy dx.
\end{aligned}$$

It is clear from this representation with the help of (4.1) that $J(f) \leq J(f_H)$.

In the case of equality, it holds that either $|A^+| = 0$ or $|S^+| = 0$. In the former case, $f = f_H$ while in the latter, $f \circ \tau_H = f_H$ \mathcal{L}^n -a.e. on B . \square

Let $\omega \in \mathbb{S}^{n-1}$. Given $0 < \tau < 1$ and $0 < \alpha \leq \pi$ the spherical cap $C_\omega(\tau, \alpha)$ is the set

$$C_\omega(\tau, \alpha) := \{x = \tau \cos \theta \omega + \tau \sin \theta \eta : 0 \leq \theta < \alpha, \eta \in \mathbb{S}^{n-1} \cap \omega^\perp\} \subset \mathbb{S}_\tau^{n-1}$$

and has volume

$$s(\tau, \alpha) := \mathcal{H}^{n-1}(S_\omega(\tau, \alpha)) = \omega_{n-2} \tau^{n-1} \int_0^\alpha (\sin \theta)^{n-2} d\theta.$$

For a Borel set E in B put

$$L(\tau) := \mathcal{H}^{n-1}(E \cap \mathbb{S}_\tau^{n-1}) \text{ for } 0 \leq \tau < 1 \text{ and } p(E) := \{0 \leq \tau < 1 : L(\tau) > 0\}.$$

The function L is Borel measurable. The spherical cap symmetrisation of E is the set

$$C_\omega E := \bigcup_{\tau \in p(E)} C_\omega(\tau, \alpha) \quad (4.2)$$

where $\alpha \in (0, \pi]$ is determined by $s(\tau, \alpha) = L(\tau)$. Observe that $C_\omega E$ is a Borel set in B (use Fubini's Theorem [1, 1.74] for example) and $|C_\omega E| = |E|$. We say that the Borel set $E \subset B$ possesses spherical cap symmetry \mathcal{L}^n -a.e. if $C_\omega E = E$ up to \mathcal{L}^n -a.e. equivalence for some $\omega \in \mathbb{S}^{n-1}$.

Let $f \in L^1_+(B)$ and choose an \mathcal{L}^n -version \tilde{f} of f . Put $m_{\tilde{f}}(\tau, t) := \mathcal{H}^{n-1}(\{\tilde{f} > t\} \cap \mathbb{S}_\tau^{n-1})$ for $t \in \mathbb{R}$ and $0 \leq \tau < 1$. The function $m_{\tilde{f}}(\tau, \cdot)$ is non-increasing and right continuous. Define its right continuous inverse by

$$\tilde{f}^\sharp(\tau, s) := \inf\{t \in \mathbb{R} : m_{\tilde{f}}(\tau, t) \leq s\} \text{ for } 0 < s \leq \mathcal{H}^{n-1}(\mathbb{S}_\tau^{n-1}).$$

For $x \in B$ put $\tau = |x|$ and choose $\alpha \in (0, \pi]$ such that $x \cdot \omega = \tau \cos \alpha$ then define

$$C_\omega \tilde{f}(x) := \tilde{f}^\sharp(\tau, s(\tau, \alpha)).$$

Note that $m_{\tilde{f}}(\tau, t) > s$ if and only if $\tilde{f}^\sharp(\tau, s) > t$. It follows that

$$\{C_\omega \tilde{f} > t\} = C_\omega \{\tilde{f} > t\} \text{ for each } t \in \mathbb{R}. \quad (4.3)$$

In particular, $C_\omega \tilde{f}$ is Borel measurable and its \mathcal{L}^n -equivalence class $C_\omega f := [C_\omega \tilde{f}]$ is the spherical cap symmetrisation of f .

Before proving Theorem 4.1, we prepare a number of lemmas. We first discuss a useful two-point inequality. We introduce the notation

$$\begin{aligned} Q &:= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0 \text{ and } x_2 \geq 0\}, \\ R &:= \{(x_1, x_2) \in Q : 0 \leq x_2 < x_1\}, \\ S &:= \{(x_1, x_2) \in Q : 0 \leq x_1 < x_2\}. \end{aligned}$$

Equip Q with the ℓ^1 -norm $\|x\|_1 := |x_1| + |x_2|$ and define a mapping $\varphi : Q \rightarrow Q$ via $(x_1, x_2) \mapsto (x_1 \vee x_2, x_1 \wedge x_2)$; φ folds S onto R . A geometric argument establishes the following lemma.

Lemma 4.3 *For any $x, y \in Q$, $\|\varphi x - \varphi y\|_1 \leq \|x - y\|_1$ with strict inequality if and only if $x \in R$ and $y \in \bar{S}$ or $x \in \bar{R}$ and $y \in S$ or the same with the rôles of x and y interchanged.*

For $\omega \in \mathbb{S}^{n-1}$ introduce the collection of closed half-spaces $\mathcal{H}_\omega := \{H_\nu : \nu \in \mathbb{S}^{n-1} \text{ and } \nu \cdot \omega \geq 0\}$.

Lemma 4.4 *Fix $\omega \in \mathbb{S}^{n-1}$. For any $H \in \mathcal{H}_\omega$ we have*

- (i) *for any $f, g \in L^1_+(B)$, $\|f_H - g_H\|_{L^1(B)} \leq \|f - g\|_{L^1(B)}$;*
- (ii) *for any $f \in L^1_+(B)$, $(C_\omega f)_H = C_\omega f$ \mathcal{L}^n -a.e. on B ;*
- (iii) *for any $f \in L^1_+(B)$,*

$$\|f_H - C_\omega f\|_{L^1(B)} \leq \|f - C_\omega f\|_{L^1(B)} \quad (4.4)$$

with strict inequality if $|\{f \circ \tau_H > f\} \cap H| > 0$.

Proof (i) By Lemma 4.3,

$$\begin{aligned}
 & \|f_H - g_H\|_{L^1(B)} \\
 &= \int_{B \cap H} |f_H - g_H| dx + \int_{B \setminus H} |f_H - g_H| dx \\
 &= \int_{B \cap H} |f \vee (f \circ \tau_H) - g \vee (g \circ \tau_H)| dx + \int_{B \setminus H} |f \wedge (f \circ \tau_H) - g \wedge (g \circ \tau_H)| dx \\
 &= \int_{B \cap H} \left\{ |f \vee (f \circ \tau_H) - g \vee (g \circ \tau_H)| + |(f \circ \tau_H) \wedge f - (g \circ \tau_H) \wedge g| \right\} dx \\
 &= \int_{B \cap H} \|\varphi(f, f \circ \tau_H) - \varphi(g, g \circ \tau_H)\|_1 dx \\
 &\leq \int_{B \cap H} \|(f, f \circ \tau_H) - (g, g \circ \tau_H)\|_1 dx \\
 &= \int_{B \cap H} \left\{ |f - g| + |f \circ \tau_H - g \circ \tau_H| \right\} dx = \|f - g\|_{L^1(B)}.
 \end{aligned}$$

(ii) Let $f \in L^1_+(B)$ and \tilde{f} a \mathcal{L}^n -representative of f . For $x \in B \cap H$, $x \cdot \omega \geq (\tau_H x) \cdot \omega$ so $(C_\omega \tilde{f})_H = C_\omega \tilde{f}$ on B . Therefore,

$$C_\omega f = [C_\omega \tilde{f}] = [(C_\omega \tilde{f})_H] = [C_\omega \tilde{f}]_H = (C_\omega f)_H.$$

(iii) The inequality follows by (i) and (ii). On $B \cap H$ the pair $(C_\omega \tilde{f}, C_\omega \tilde{f} \circ \tau_H)$ belongs to \bar{R} . By Lemma 4.3 if $(\tilde{f}, \tilde{f} \circ \tau_H) \in S$ on a set of positive measure in $B \cap H$ then strict inequality holds in (4.4). This observation leads to the criterion in the lemma. \square

Lemma 4.5 Let $f \in L^1(\mathbb{S}^{n-1}, \mathcal{H}^{n-1})$. Fix $v \in \mathbb{S}^{n-1}$ and let (v_h) be a sequence in \mathbb{S}^{n-1} that converges to v in \mathbb{S}^{n-1} . Then

$$\|f \circ \tau_h - f \circ \tau\|_{L^1(\mathbb{S}^{n-1}, \mathcal{H}^{n-1})} \rightarrow 0 \text{ as } h \rightarrow \infty.$$

Proof Note that $|\tau_h x - \tau x| \leq 4|v_h - v|$ for each $x \in \mathbb{S}^{n-1}$ and h . Now use the density of $C(\mathbb{S}^{n-1})$ in $L^1(\mathbb{S}^{n-1}, \mathcal{H}^{n-1})$ and the fact that each τ, τ_h is an isometry on $L^1(\mathbb{S}^{n-1}, \mathcal{H}^{n-1})$. \square

The next lemma is a spherical cap symmetrisation counterpart to [4, Lemma 6.3] and extends [23, Lemma 3.9].

Lemma 4.6 Let $f \in L^1_+(B)$ and $\omega \in \mathbb{S}^{n-1}$ and assume that $f \neq C_\omega f$. Then there exists $H \in \mathcal{H}_\omega$ such that

$$\|f_H - C_\omega f\|_{L^1(B)} < \|f - C_\omega f\|_{L^1(B)}.$$

Proof For non-negative Borel measurable functions f, g on B , $f = g$ if and only if $\{|f > t\} \Delta \{g > t\} = \emptyset$ for any $t > 0$. As $f \neq C_\omega f$ there exists $t > 0$ such that

$$|\{f > t\} \Delta \{C_\omega f > t\}| > 0.$$

By (4.3) $|\{f > t\}| = |\{C_\omega f > t\}|$ and it follows that $|\{f \leq t < C_\omega f\}| = |\{C_\omega f \leq t < f\}|$. Put $A := \{f \leq t < C_\omega f\}$ and $A' := \{C_\omega f \leq t < f\}$. For later use we note that

$$\mathcal{H}^{n-1}(A \cap \mathbb{S}_\tau^{n-1}) = \mathcal{H}^{n-1}(A' \cap \mathbb{S}_\tau^{n-1})$$

for \mathcal{L}^1 -a.e. $\tau \in (0, 1)$.

We claim there exists $H \in \mathcal{H}_\omega$ such that $|A \cap \tau_H A'| > 0$. Taking this as read, on $A \cap \tau_H A'$ we have that $C_\omega f > t \geq C_\omega f \circ \tau_H$ so that $A \cap \tau_H A' \subset H$. Also, $f \leq t < f \circ \tau_H$ there. In short, $A \cap \tau_H A' \subset \{f \circ \tau_H > f\} \cap H$. So $|\{f \circ \tau_H > f\} \cap H| > 0$ and strict inequality holds by Lemma 4.4 (iii).

To prove the claim, assume for a contradiction that $|A \cap \tau_H A'| = 0$ for all $H \in \mathcal{H}_\omega$. Let F be a countable dense subset in $\mathbb{S}^{n-1} \cap H_\omega$. Then

$$\left| \bigcup_{v \in F} (A \cap \tau_{H_v} A') \right| = 0.$$

Therefore for all $r \in (0, 1)$ it holds that

$$\mathcal{H}^{n-1}(A_r \cap \tau_{H_v} A'_r) = 0 \quad \text{for every } v \in F,$$

except on a \mathcal{L}^1 -null set $N \subset (0, 1)$. We write $A_r := A \cap \mathbb{S}_r^{n-1}$ for the r -section of A and likewise for A' . Let $v \in \mathbb{S}^{n-1} \cap H_\omega$ with corresponding reflection $\tau = \tau_{H_v}$. Select a sequence (v_h) in F which converges to v in \mathbb{S}^{n-1} and write τ_h for the reflection associated to the closed half-space H_{v_h} . For $r \in (0, 1) \setminus N$,

$$|\mathcal{H}^{n-1}(A_r \cap \tau A'_r) - \mathcal{H}^{n-1}(A_r \cap \tau_h A'_r)| \leq \|\chi_{A'} - \chi_{A'} \circ \tau \circ \tau_h\|_{L^1(\mathbb{S}_r^{n-1}, \mathcal{H}^{n-1})},$$

and this latter converges to zero as $h \rightarrow \infty$ by Lemma 4.5. We derive that

$$\mathcal{H}^{n-1}(A_r \cap \tau_{H_v} A'_r) = 0 \quad \text{for every } v \in \mathbb{S}^{n-1} \cap H_\omega \quad (4.5)$$

for all $r \in (0, 1) \setminus N$.

To conclude the argument, choose $r \in (0, 1) \setminus N$ such that $\mathcal{H}^{n-1}(A_r) = \mathcal{H}^{n-1}(A'_r) > 0$. Select a density point x for A_r lying in A_r using [1, Corollary 2.23] for example; that is,

$$\lim_{\rho \downarrow 0} \frac{1}{\mathcal{H}^{n-1}(B(x, \rho)_r)} \int_{B(x, \rho)_r} |\chi_{A_r}(z) - \chi_{A_r}(x)| \mathcal{H}^{n-1}(dz) = 0.$$

This means that A_r has density 1 at x in the sense that

$$\frac{\mathcal{H}^{n-1}(A_r \cap B(x, \rho))}{\mathcal{H}^{n-1}(\mathbb{S}_r^{n-1} \cap B(x, \rho))} \rightarrow 1 \quad \text{as } \rho \downarrow 0.$$

Choose y in A'_r similarly so that A'_r has density 1 at y . Then $C_\omega f(x) > t \geq C_\omega f(y)$. So there exists $v \in \mathbb{S}^{n-1} \cap H_\omega$ such that with $\tau = \tau_{H_v}$ we have that $\tau y = x$. But then

$$\lim_{\rho \downarrow 0} \frac{\mathcal{H}^{n-1}(A_r \cap \tau B_r \cap B(x, \rho))}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1} \cap B(x, \rho))} = 1,$$

so that, in fact, $\mathcal{H}^{n-1}(A_r \cap \tau B_r) > 0$, contradicting (4.5). \square

Proof of Theorem 4.1 Let E be an optimal configuration for the data (U, p) . Assume for a contradiction that $E \neq C_\omega E$ \mathcal{L}^n -a.e. for any $\omega \in \mathbb{S}^{n-1}$. Then there exists $\omega \in \mathbb{S}^{n-1}$ such that

$$\delta := \inf_{v \in \mathbb{S}^{n-1}} \|\chi_E - C_v \chi_E\|_{L^1(B)} = \|\chi_E - C_\omega \chi_E\|_{L^1(B)} > 0.$$

By Lemma 4.6 there exists $H \in \mathcal{H}_\omega$ such that $\|(\chi_E)_H - C_\omega \chi_E\|_{L^1(B)} < \|\chi_E - C_\omega \chi_E\|_{L^1(B)}$. It is plain from this that $\chi_E \neq (\chi_E)_H$; but also $\chi_E \circ \tau_H \neq (\chi_E)_H$, for otherwise,

$$\begin{aligned} \|\chi_E - C_{\tau_H \omega} \chi_E\|_{L^1(B)} &= \|\chi_E - C_\omega \chi_E \circ \tau_H\|_{L^1(B)} = \|(\chi_E)_H - C_\omega \chi_E\|_{L^1(B)} \\ &< \|\chi_E - C_\omega \chi_E\|_{L^1(B)}, \end{aligned}$$

contradicting optimality of ω . It follows by Theorem 4.2 that $J(E) < J(E_H)$, contradicting the fact that E is an optimal configuration for the data (U, p) . The result now follows. \square

5 Preliminaries on Weighted Dirichlet Forms

Let $n = 1$ and $U = (a, b)$ be an open bounded interval in \mathbb{R} . We are given a density function w with the property

(A) w is a positive function in $C_0(U)$.

The weighted volume of an \mathcal{L}^1 -measurable set E in U is given by $m(E) := \int_E w \, dx$. We introduce the further assumption

(B) $w \in C^1(U)$ and $w'/w \in L^2(U, m)$.

Consider the coercive bilinear form

$$\mathcal{E}(u, v) := \int_U (uv + u'v') \, dm \quad (u, v \in \mathcal{D} := C^\infty(\overline{U}))$$

in $L^2(U, m)$.

Lemma 5.1 Assume (A)–(B). Then

- (i) $(\mathcal{D}, \mathcal{E})$ is closable in $L^2(U, m)$ with closure denoted $(D(\mathcal{E}), \mathcal{E})$;
- (ii) $(D(\mathcal{E}), \mathcal{E})$ is a symmetric Dirichlet form in $L^2(U, m)$.

Proof We refer to [19, Definitions I.2.3 and I.4.5] (for example). Note that $(\mathcal{D}, \mathcal{E})$ satisfies the weak sector condition [19] (2.3) by the Cauchy–Schwarz inequality. Suppose (u_h) is a sequence in \mathcal{D} such that $u_h \rightarrow 0$ in $L^2(U, m)$. For $v \in \mathcal{D}$ an integration-by-parts gives

$$\begin{aligned} \int_U u'_h v' w \, dx &= \int_{\partial U} u_h v' w \nu \, d\mathcal{H}^0 - \int_U u_h (v' w)' \, dx \\ &= - \int_U u_h \frac{(v' w)'}{w} \, dm = - \int_U u_h \left\{ v'' + v' \frac{w'}{w} \right\} \, dm \rightarrow 0 \end{aligned}$$

as $h \rightarrow \infty$ where $\nu = \pm 1$ is the one-dimensional unit exterior normal on ∂U . We have made use of the assumptions (A) and (B). The statement (i) follows by [19, Lemma I.3.4]. Then $(D(\mathcal{E}), \mathcal{E})$ is a symmetric closed form by definition (cf. [19, Definition I.2.3]). By [19, Proposition I.4.10 and II.2 (c)], $(D(\mathcal{E}), \mathcal{E})$ is a symmetric Dirichlet form. \square

Given a real-valued function u on \mathbb{R}_+ (or \mathbb{R}) define the function $\theta_t u$ on \mathbb{R}_+ for each $t > 0$ by $(\theta_t u)(x) := u(x + t)$ for $x \in \mathbb{R}_+$ (or \mathbb{R}).

Lemma 5.2 *Let λ be a positive \mathcal{L}^1 -integrable function on \mathbb{R}_+ such that*

$$c := \sup_{t>0} \sup_{x>0} \frac{\lambda(x)}{\lambda(x+t)} < \infty.$$

Then

- (i) $\theta_t \in B(L^2(\mathbb{R}_+, \lambda \mathcal{L}^1))$ for each $t > 0$;
- (ii) $\|u - \theta_t u\|_{L^2(\mathbb{R}_+, \lambda \mathcal{L}^1)} \rightarrow 0$ as $t \downarrow 0$ for each $u \in L^2(\mathbb{R}_+, \lambda \mathcal{L}^1)$.

Proof (i) For any $u \in L^2(\mathbb{R}_+, \lambda \mathcal{L}^1)$,

$$\begin{aligned} \|\theta_t u\|_{L^2(\mathbb{R}_+, \lambda \mathcal{L}^1)}^2 &= \int_0^\infty |\theta_t u|^2 \lambda \, dx \leq c \int_0^\infty |u(x+t)|^2 \lambda(x+t) \, dx \\ &\leq c \|u\|_{L^2(\mathbb{R}_+, \lambda \mathcal{L}^1)}^2. \end{aligned}$$

(ii) The statement holds for $u \in C([0, \infty)) \cap L^2(\mathbb{R}_+, \lambda \mathcal{L}^1)$ by the dominated convergence theorem and this latter set is dense in $L^2(\mathbb{R}_+, \lambda \mathcal{L}^1)$. These observations as well as (i) lead to the result using a 3ε -argument. \square

Our next assumption is stronger than required but easy to state:

(C) w is unimodal on U .

Lemma 5.3 *Assume (A)–(C). Then*

$$D(\mathcal{E}) = \left\{ u \in L^2(U, m) : u \text{ is weakly differentiable on } U \text{ and } u' \in L^2(U, m) \right\}.$$

Proof Let $u \in D(\mathcal{E})$. There exists a Cauchy sequence (u_h) in $(\mathcal{D}, \mathcal{E})$ which converges to u in $L^2(U, m)$. Then (u'_h) is a Cauchy sequence in $L^2(U, m)$ with limit $v \in L^2(U, m)$ (say). For $\phi \in C_c^\infty(U)$,

$$\begin{aligned} \int_U u \phi' dx &= \lim_h \int_U u_h \phi' dx = - \lim_h \int_U u'_h \phi dx = - \lim_h \int_U u'_h (w^{-1} \phi) dm \\ &= - \int_U v \phi dx; \end{aligned}$$

so u is weakly differentiable on U with weak derivative $u' = v \in L^2(U, m)$. Now let $u \in L^2(U, m)$ be weakly differentiable on U such that $u' \in L^2(U, m)$. Multiplying by a partition of unity we may assume that $u = 0$ near b . Denote by \bar{u} the extension of u to \mathbb{R} by zero. For $t > 0$ put $v_t := (\theta_t \bar{u})|_U$. Note that v_t is weakly differentiable and $v'_t = \theta_t(\bar{u}')|_U$. For $t > 0$, $v_t, v'_t \in L^2(U, m)$. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be a family of mollifiers on \mathbb{R} (cf. [1, 2.1]). For $t > 0$ and $\varepsilon > 0$ small, $(\rho_\varepsilon \star (\theta_t \bar{u}))' = \rho_\varepsilon \star (\theta_t \bar{u}')$ on U . The operation \star stands for convolution. For $t > 0$ and $\varepsilon > 0$ small put $w_{t,\varepsilon} := \rho_\varepsilon \star (\theta_t \bar{u})|_U \in C^\infty(\bar{U})$. Now

$$\begin{aligned} \|u - w_{t,\varepsilon}\|_{L^2(U,m)} &\leq \|u - v_t\|_{L^2(U,m)} + \|v_t - w_{t,\varepsilon}\|_{L^2(U,m)}; \\ \|u' - w'_{t,\varepsilon}\|_{L^2(U,m)} &\leq \|u' - v'_t\|_{L^2(U,m)} + \|v'_t - \rho_\varepsilon \star (\theta_t \bar{u}')\|_{L^2(U,m)}. \end{aligned}$$

By Lemma 5.2 and (A)–(C) the expressions $\|u - v_t\|_{L^2(U,m)}$ and $\|u' - v'_t\|_{L^2(U,m)}$ are small for $t > 0$ small. We also use the fact that the mollified functions are regular approximations in L^2 (cf. [1, 2.1]). This shows that $u \in D(\mathcal{E})$. \square

Suppose that $\hat{U} = (c, d)$ is an open bounded interval in \mathbb{R} and $\Phi : \hat{U} \rightarrow U$ is a C^1 bijection such that $\Phi' =: \varphi > 0$ on \hat{U} . Let $\hat{m} := \Phi_\# m$ be the pull-back of m under Φ ; thus $\hat{m} = \hat{w} \mathcal{L}^1$ on \hat{U} where $\hat{w} := \varphi(w \circ \Phi)$. Define a coercive bilinear form

$$\hat{\mathcal{E}}(u, v) := \int_{\hat{U}} (uv + \varphi^{-2} u' v') d\hat{m} \quad (u, v \in D(\hat{\mathcal{E}}))$$

in $L^2(\hat{U}, \hat{m})$ with domain

$$D(\hat{\mathcal{E}}) := \left\{ u \in L^2(\hat{U}, \hat{m}) : u \text{ is weakly differentiable on } \hat{U} \text{ and } \varphi^{-1} u' \in L^2(\hat{U}, \hat{m}) \right\}.$$

Lemma 5.4 Assume (A)–(C). Then

- (i) $(D(\hat{\mathcal{E}}), \hat{\mathcal{E}})$ is a symmetric Dirichlet form in $L^2(\hat{U}, \hat{m})$;
- (ii) the mapping $D(\hat{\mathcal{E}}) \rightarrow D(\mathcal{E}); u \mapsto \bar{u} := u \circ \Phi^{-1}$ is a Hilbert space isomorphism.

Proof (i) We show that $(D(\hat{\mathcal{E}}), \hat{\mathcal{E}})$ is closed in $L^2(\hat{U}, \hat{m})$. Let (u_h) be an $\hat{\mathcal{E}}^{1/2}$ -Cauchy sequence in $D(\hat{\mathcal{E}})$. Then (u_h) resp. $(\varphi^{-1} u'_h)$ are Cauchy sequences in $L^2(\hat{U}, \hat{m})$ with

limits $u \in L^2(\hat{U}, \hat{m})$ resp. $v \in L^2(\hat{U}, \hat{m})$. For $\phi \in C_c^\infty(\hat{U})$,

$$\begin{aligned} \int_{\hat{U}} u \phi' dx &= \lim_{h \rightarrow \infty} \int_{\hat{U}} u_h \phi' dx = - \lim_{h \rightarrow \infty} \int_{\hat{U}} u'_h \phi dx \\ &= - \lim_{h \rightarrow \infty} \int_{\hat{U}} \varphi^{-1} u'_h \frac{\phi}{w \circ \Phi} d\hat{m} = - \int_{\hat{U}} v \frac{\phi}{w \circ \Phi} d\hat{m} = - \int_{\hat{U}} (\varphi v) \phi dx \end{aligned}$$

so u is weakly differentiable and $\varphi^{-1}u' = v$; that is, $u \in D(\hat{\mathcal{E}})$. It then follows that (u_h) converges to u in $\hat{\mathcal{E}}^{1/2}$ -norm. So $(D(\hat{\mathcal{E}}), \hat{\mathcal{E}})$ is a symmetric closed form in $L^2(\hat{U}, \hat{m})$.

Let $u \in D(\hat{\mathcal{E}})$. Given $\varepsilon > 0$ let φ_ε be as in [19, Example II.2.7]. Note that $u \in W_{\text{loc}}^{1,2}(\hat{U})$. Then $\varphi_\varepsilon(u)$ is weakly differentiable on \hat{U} and $\varphi_\varepsilon(u)' = \varphi'_\varepsilon(u)u'$ (see for example [10, 5.10, Exercise 16]) so $\varphi_\varepsilon(u) \in D(\hat{\mathcal{E}})$. We then derive that

$$\hat{\mathcal{E}}(\varphi_\varepsilon(u), \varphi_\varepsilon(u)) \leq \hat{\mathcal{E}}(u, u).$$

By [19, Proposition I.4.7], $(D(\hat{\mathcal{E}}), \hat{\mathcal{E}})$ is a symmetric Dirichlet form in $L^2(\hat{U}, \hat{m})$.

(ii) Let $u \in D(\hat{\mathcal{E}})$. Note that $(u'/\varphi) \circ \Phi^{-1} \in L^2(U, m)$ (use [1, (2.47)]) and \bar{u} is weakly differentiable on U with $\bar{u}' = (u'/\varphi) \circ \Phi^{-1} \in L^2(U, m)$. Thus the mapping is well-defined. For $u, v \in D(\hat{\mathcal{E}})$, $\hat{\mathcal{E}}(u, v) = \mathcal{E}(\bar{u}, \bar{v})$ again using [1, (2.47)]. In particular, the mapping $u \mapsto \bar{u}$ is injective. Now let $u \in D(\mathcal{E})$ and put $\hat{u} := u \circ \Phi$. Then $\hat{u} \in L^2(\hat{U}, \hat{m})$, \hat{u} is weakly differentiable on \hat{U} with weak derivative $\hat{u}' = \varphi(u' \circ \Phi)$ and $\varphi^{-1}\hat{u}' \in L^2(\hat{U}, \hat{m})$; in other words, $\hat{u} \in D(\hat{\mathcal{E}})$. This shows that the mapping in (ii) is surjective. \square

Now take $B = (-1, 1)$. We are given density functions f, g with the properties

- (A.1) f is a positive function in $C(B)$;
- (A.2) g is a positive unimodal function in $C_0(B)$.

The weighted volume of an \mathcal{L}^1 -measurable set E in B is given by $V(E) := \int_E f dx$. Put $\rho := f/g$. We introduce the further assumption

- (A.3) $\rho \in L^1(B, \mathcal{L}^1)$, $g \in C^1(B)$ and $g'/f \in L^2(B, V)$.

Define

$$R : B \rightarrow \mathbb{R}; t \mapsto \int_0^t \rho(\tau) d\tau,$$

and let \check{B} denote the image of B under R ; \check{B} is an open bounded interval in \mathbb{R} . Then $R : B \rightarrow \check{B}$ is a C^1 bijection. Define $\check{g} := g \circ R^{-1}$ on \check{B} . Define the measure $\check{V} := \check{g}\mathcal{L}^1$ on \check{B} .

We introduce coercive bilinear forms

$$\check{\mathcal{E}}(u, v) := \int_{\check{B}} (uv + u'v') d\check{V} \quad (u, v \in D(\check{\mathcal{E}}))$$

in $L^2(\check{B}, \check{V})$ with domain

$$D(\check{\mathcal{E}}) = \left\{ u \in L^2(\check{B}, \check{V}) : u \text{ is weakly differentiable on } \check{B} \text{ and } u' \in L^2(\check{B}, \check{V}) \right\};$$

and

$$\hat{\mathcal{E}}(u, v) := \int_B \left(uv + \rho^{-2} u' v' \right) dV \quad (u, v \in D(\hat{\mathcal{E}}))$$

in $L^2(B, V)$ with domain

$$D(\hat{\mathcal{E}}) = \left\{ u \in L^2(B, V) : u \text{ is weakly differentiable on } B \text{ and } \rho^{-1} u' \in L^2(B, V) \right\}.$$

Note that with \check{B} in place of U , $w := \check{g}$ satisfies properties (A)–(C) above in light of the assumptions (A.1)–(A.3). We derive

Lemma 5.5 *Assume (A.1)–(A.3). Then*

- (i) $(D(\check{\mathcal{E}}), \check{\mathcal{E}})$ is a symmetric Dirichlet form in $L^2(\check{B}, \check{V})$;
- (ii) $C^\infty(\check{B})$ is dense in $D(\check{\mathcal{E}})$ with respect to the $\check{\mathcal{E}}^{1/2}$ -norm;
- (iii) $(D(\hat{\mathcal{E}}), \hat{\mathcal{E}})$ is a symmetric Dirichlet form in $L^2(B, V)$;
- (iv) the mapping $D(\hat{\mathcal{E}}) \rightarrow D(\check{\mathcal{E}}); u \mapsto u \circ R^{-1}$ is a Hilbert space isomorphism.

6 An (f, g) -Isoperimetric Inequality

Recall that an \mathcal{L}^1 -measurable set $E \subset B$ is said to be a Caccioppoli set if for each relatively compact open set Ω in B ,

$$P(E, \Omega) := \sup \left\{ \int_\Omega \chi_E \phi' dx : \phi \in C_c^\infty(\Omega, \mathbb{R}), \|\phi\|_\infty \leq 1 \right\} < \infty.$$

There then exists a unique real Radon measure $D\chi_E$ on B such that

$$\int_B \chi_E \phi' dx = - \int_B \phi dD\chi_E$$

for all $\phi \in C_c^\infty(B, \mathbb{R})$ [1, Corollary 1.55]. Denote by $|D\chi_E|$ the total variation measure of $D\chi_E$.

Theorem 6.1 *Suppose that E is a Caccioppoli set in B with $|E| > 0$. Then there exist $N \in \mathbb{N} \cup \{\infty\}$ and closed intervals $E_h = [a_{2h-1}, a_{2h}] \subset \mathbb{R}$ ($h = 1, \dots, N$) with non-empty interior and separated by open neighbourhoods in \mathbb{R} such that E is \mathcal{L}^1 -a.e. equivalent to the union of the E_h .*

The statement that the collection of intervals (E_h) is separated by open intervals means that $\inf_{k \neq h} d(E_h, E_k) > 0$ for each h . Here, d denotes the standard metric on \mathbb{R} .

Proof The proof is along the lines of [1, Proposition 3.52]. Put $u := \chi_E \in \text{BV}_{\text{loc}}(B)$. Then $\mu := Du$ is a real Radon measure on B [1, 1.40]. Define

$$w(t) := \begin{cases} -\mu([t, 0)) & \text{for } -1 < t < 0; \\ 0 & \text{for } t = 0; \\ \mu([0, t)) & \text{for } 0 < t < 1; \end{cases}$$

w is left-continuous on B by the Hahn decomposition and inner/outer regularity [1, 1.43]. By Fubini's theorem, $w \in \text{BV}_{\text{loc}}(B)$ and $Dw = \mu$. By [1, Proposition 3.2], $u = c + w$ \mathcal{L}^1 -a.e. on B for some $c \in \mathbb{R}$. Let A be the set of atoms of μ in B . Note that w is continuous on $B \setminus A$ and $w(t+) - w(t) = \mu(\{t\})$ at each $t \in A$. As $c + w \in \{0, 1\}$ \mathcal{L}^1 -a.e. on B , $\mu(\{t\}) \in \{-1, 1\}$ for each $t \in A$. Let Ω be a relatively compact open set in B . Then $\text{Card}(A \cap \Omega) = |\mu|(A \cap \Omega) \leq |\mu|(\Omega) < \infty$. Thus the set of atoms A accumulates at ∂B (if at all). By the observations above, the function $c + w$ is constant on each connected component of $B \setminus A$ with values in the set $\{0, 1\}$. Let the sets E_h be the closure of the open intervals in $B \setminus A$ where $c + w$ takes the value 1. \square

Let g be a positive lower semicontinuous function on B . Let E be a Caccioppoli set in B . The g -perimeter of E relative to B is defined by

$$P_g(E, B) := \int_B g \, d|D\chi_E|. \quad (6.1)$$

Lemma 6.2 *Let g be a positive lower semicontinuous function on B and E a Caccioppoli set in B . Then*

$$P_g(E, B) = \sum_{a \in A} g(a)$$

where A stands for the set of atoms of $D\chi_E$ in B .

Proof A direct computation gives $D\chi_E = \sum_{a \in A} D\chi_E(\{a\})\delta_a$ and $|D\chi_E| = \sum_{a \in A} \delta_a$ which gives the result. We use δ_a to stand for the Dirac measure at a . \square

Let f, g be densities on B satisfying conditions (A.1)–(A.3). The weighted volume of an \mathcal{L}^1 -measurable set E in B is the measure given by $V(E) := \int_E f \, dx$. Define $F : [-1, 1] \rightarrow [0, V(B)]$ by $F(x) := V((-1, x))$ and

$$J(p) := (g \circ F^{-1})(p) \text{ for } p \in [0, V(B)].$$

We impose the additional assumptions

$$(A.4) \quad J(p) = J(V(B) - p) \text{ for } 0 < p < V(B);$$

$$(A.5) \quad \text{for all } p, q > 0 \text{ with } p + q < V(B), \quad J(p + q) < J(p) + J(q).$$

For an \mathcal{L}^1 -measurable set E in B the \star -rearrangement of E is defined by $E^\star := (-1, F^{-1}(V(E)))$. We then have that the following (f, g) -isoperimetric inequality is valid.

Theorem 6.3 Assume (A.1)–(A.5). Suppose that E is a Caccioppoli set in B . Then $P_g(E, B) \geq P_g(E^\star, B)$.

Proof For $0 < p < q < V(B)$,

$$J(p) + J(q) = J(p) + J(V(B) - q) > J(p + V(B) - q) = J(q - p) \quad (6.2)$$

by (A.4)–(A.5). The above inequality also holds for $0 \leq p < q \leq V(B)$ with $>$ replaced by \geq by continuity; equality holds if and only if one or both of p, q are extremal.

We may suppose that $V(E) = p$ for some $p \in (0, V(B))$. Assume that E has the form $E = \bigcup_{h=1}^N [a_{2h-1}, a_{2h}]$ with $-1 \leq a_1 < a_2 < \dots < a_{2N-1} < a_{2N} \leq 1$ for some $N \in \mathbb{N}$. Put $p_j := F(a_j)$ so that $0 \leq p_1 < \dots < p_{2N} \leq V(B)$ and $\sum_{h=1}^N (p_{2h} - p_{2h-1}) = p$. By Lemma 6.2, (6.2) and (A.5),

$$\begin{aligned} P_g(E, B) &= \sum_{h=1}^N \left(J(p_{2h-1}) + J(p_{2h}) \right) \geq \sum_{h=1}^N J(p_{2h} - p_{2h-1}) \\ &\geq J\left(\sum_{h=1}^N (p_{2h} - p_{2h-1})\right) = J(p) = P_g(E^\star, B). \end{aligned}$$

The result for arbitrary E as in the statement follows by Theorem 6.1, the monotone convergence theorem and continuity of J . \square

We now investigate the equality case in the isoperimetric inequality.

Theorem 6.4 Assume (A.1)–(A.5). Suppose that E is a Caccioppoli set in B . Assume that $P_g(E, B) = P_g(E^\star, B)$. Then either $E = E^\star$ or $B \setminus E = (B \setminus E)^\star$ V -a.e.

Proof We may assume that E is the union of closed intervals $E_h \subset \mathbb{R}$ ($h = 1, \dots, N, N \in \mathbb{N} \cup \{\infty\}$) with non-empty interior and separated by open neighbourhoods in \mathbb{R} as in Theorem 6.1. In virtue of (6.1) we may take $N > 1$. We can then find $x \in B \setminus E$ such that $V(E \cap (x, 1)) > 0$ and $V(E \cap (-1, x)) > 0$. Put $E_- := E \cap (-1, x)$, $E_+ := E \cap (x, 1)$, $F_- := E_-^\star$ and $F_+ := B \setminus (B \setminus E_+)^\star$. Note that $V(E_-) = V(F_-) = p_-$ and $V(E_+) = V(F_+) = p_+$ for some $p_\pm \in (0, V(B))$. We have

$$\begin{aligned} J(p_-) + J(p_+) &= P_g(F_-, B) + P_g(F_+, B) \leq P_g(E_-, B) + P_g(E_+, B) \\ &= P_g(E, B) \end{aligned}$$

by Theorem 6.3. On the other hand, $P_g(E, B) = P_g(E^\star, B) = J(p_- + p_+) < J(p_-) + J(p_+)$ by (A.5), a contradiction. \square

Proposition 6.5 Assume (A.1)–(A.3). If g'/f is strictly decreasing on B then (A.5) holds.

Proof Note that J is differentiable on $(0, V(B))$. Moreover, $(F^{-1})'(p) = 1/(f \circ F^{-1})(p)$ and

$$J'(p) = g'(F^{-1}(p)) \cdot 1/(f \circ F^{-1})(p) = (g'/f)(F^{-1}(p))$$

for $p \in (0, V(B))$; this shows that J is strictly concave on $(0, V(B))$ as $(0, V(B)) \mapsto (-1, 1) : p \mapsto (F^{-1})(p)$ is increasing. For $0 < p < q < V(B)$ we have that

$$\frac{J(p)}{p} \geq \frac{J(q)}{q} \quad (6.3)$$

by concavity and the fact that $J(0) \geq 0$. Supposing additionally that $p + q < V(B)$ we have

$$\frac{J(q) - J(p)}{q - p} > \frac{J(p + q) - J(q)}{p}$$

by considering the gradient of chords. Rearranging and using (6.3) gives (A.5). \square

Let u be a real-valued \mathcal{L}^1 -measurable function on B . Put $\mu_u(t) := V(\{|u| > t\})$ for $t \geq 0$. The function $\mu_u : [0, \infty) \rightarrow [0, V(B)]$ is non-increasing, right-continuous and $\mu_u(t) \rightarrow 0$ as $t \rightarrow \infty$. Define its right-continuous inverse $u^\sharp : [0, V(B)] \rightarrow [0, \infty]$ by

$$u^\sharp(s) := \inf\{t \geq 0 : \mu_u(t) \leq s\} \text{ for } 0 \leq s \leq V(B),$$

with the understanding that $\inf \emptyset = +\infty$. Define $u^\star := u^\sharp \circ F$ on B . Note that $\mu_u(t) > s$ if and only if $u^\sharp(s) > t$ (see Lemma 10.1). It follows that

$$V(\{|u| > t\}) = V(\{u^\star > t\}) \text{ for each } t \geq 0; \quad (6.4)$$

in fact, $V(\{u^\star > t\}) = V(\{F < \mu_u(t)\}) = V((-1, F^{-1}(\mu_u(t)))) = \mu_u(t)$.

The following result is a Hardy–Littlewood type inequality and can be proved as in [14, 13.10] (see also [9, Theorem 3]).

Theorem 6.6 Assume (A.1)–(A.5). Let u, v be real-valued \mathcal{L}^1 -measurable functions on B . Then

$$\int_B |uv| dV \leq \int_B u^\star v^\star dV.$$

The next non-expansivity result can be found in [9, Corollary 1].

Theorem 6.7 Assume (A.1)–(A.5). Let u, v be real-valued \mathcal{L}^1 -measurable functions on B . Then

$$\int_B |u^\star - v^\star|^2 dV \leq \int_B |u - v|^2 dV.$$

Define a metric \hat{d} on B as follows. The length of a piecewise C^1 parametrised curve $\gamma : [\alpha, \beta] \rightarrow B$ in (B, \hat{d}) is

$$\hat{L}[\gamma] = \int_{\alpha}^{\beta} \rho(\gamma(t)) |\dot{\gamma}(t)| dt.$$

For $x, y \in B$, $\hat{d}(x, y)$ stands for the infimum of lengths of piecewise C^1 parametrised curves in B connecting x to y ; $\hat{d}(\cdot, \cdot)$ is a metric on B .

Lemma 6.8 For $x, y \in B$, $\hat{d}(x, y) = d(R(x), R(y))$.

Proof Let $\gamma : [\alpha, \beta] \rightarrow B$ be a piecewise C^1 parametrised curve in B connecting x to y . Then $\hat{L}[\gamma] = L[R \circ \gamma]$ in an obvious notation. So $\hat{d}(x, y) \leq d(R(x), R(y))$. A similar argument gives the reverse inequality. \square

Note that for each \mathcal{L}^1 -measurable set E in B , $\check{V}(R(E)) = V(E)$.

Lemma 6.9 The mapping $R : B \rightarrow \check{B}$ sets up a one-to-one correspondence between Caccioppoli sets in B resp. \check{B} . Moreover, let E be a Caccioppoli set in B . Then

- (i) $|D\chi_{R(E)}| = R_{\#}|D\chi_E|$;
- (ii) $P_g(R(E), \check{B}) = P_g(E, B)$.

Proof Let $\Omega \subset B$ be a relatively compact open set. Then

$$\begin{aligned} |D\chi_{R(E)}|(R(\Omega)) &= \sup \left\{ \int_{R(\Omega)} \chi_{R(E)} \phi' dx : \phi \in C_c^1(\check{B}, \mathbb{R}) \text{ and } \|\phi\|_{\infty} \leq 1 \right\} \\ &= \sup \left\{ \int_{\Omega} \chi_E (\phi \circ R)' dx : \phi \in C_c^1(\check{B}, \mathbb{R}) \text{ and } \|\phi\|_{\infty} \leq 1 \right\} \\ &= \sup \left\{ \int_{\Omega} \chi_E \phi' dx : \phi \in C_c^1(B, \mathbb{R}) \text{ and } \|\phi\|_{\infty} \leq 1 \right\} \\ &= |D\chi_E|(\Omega), \end{aligned}$$

from which the first assertion follows. Item (i) follows from the definition of the push-forward [1, Definition 1.7] and the coincidence criterion [1, Proposition 1.8], while (ii) follows from (6.1), (i) and the change of variables formula for integrals. \square

The function $\check{F} : \mathbb{R} \rightarrow [0, V(B)]$ defined by $\check{F}(x) := \check{V}(\check{B} \cap (-\infty, x))$ is the cumulative distribution function of \check{g} . Let u be a real-valued \mathcal{L}^1 -measurable function on \check{B} . Put $\check{\mu}_u(t) := \check{V}(\{|u| > t\})$ for $t \geq 0$ and denote by $\check{u}^{\sharp} : [0, \check{V}(\check{B})] \rightarrow [0, \infty]$ its right-continuous inverse (as in the Appendix). Define $u^{\star} := \check{u}^{\sharp} \circ \check{F}$ on \check{B} .

Proposition 6.10 Let u be a real-valued \mathcal{L}^1 -measurable function on B and put $v := u \circ R^{-1}$. Then $u^{\star} = v^{\star} \circ R$. In particular, for any \mathcal{L}^1 -measurable set $E \subset B$, $R(E)^{\star} = R(E^{\star})$.

Proof We have that $\mu_u(t) = \check{\mu}_v(t)$ for each $t \geq 0$; hence $u^{\sharp} = \check{v}^{\sharp}$ on $[0, V(B)]$. Now, $\check{F} \circ R = F$ on $[-1, 1]$. This leads to the first claim. The second then follows straightforwardly. \square

Corollary 6.11 Assume (A.1)–(A.5). Suppose that E is a Caccioppoli set in \check{B} . Then $P_{\check{g}}(E, \check{B}) \geq P_{\check{g}}(E^*, \check{B})$.

Proof Let E be a Caccioppoli set in B . By Lemma 6.9, Theorem 6.3 and Proposition 6.10,

$$P_{\check{g}}(R(E), \check{B}) = P_g(E, B) \geq P_g(E^\star, B) = P_{\check{g}}(R(E^\star), \check{B}) = P_{\check{g}}(R(E)^\star, \check{B}).$$

□

Corollary 6.12 Assume (A.1)–(A.5). Suppose that E is a Caccioppoli set in \check{B} . Assume that $P_{\check{g}}(E, \check{B}) = P_{\check{g}}(E^*, \check{B})$. Then either $E = E^*$ or $\check{B} \setminus E = (\check{B} \setminus E)^\star \check{V}$ -a.e.

Proof This follows from Lemma 6.9 and Theorem 6.4. □

Finally, we state a counterpart of Theorem 6.7.

Theorem 6.13 Assume (A.1)–(A.5). Let u, v be real-valued \mathcal{L}^1 -measurable functions on \check{B} . Then

$$\int_{\check{B}} |u^\star - v^\star|^2 d\check{V} \leq \int_{\check{B}} |u - v|^2 d\check{V}.$$

7 A Pólya–Szegő Inequality

We first show that the rearrangement \cdot^\star is smoothing in the sense of [21] (see also [4]). Given $r > 0$ write E_r for the r -neighbourhood of an \mathcal{L}^1 -measurable set E in (\check{B}, d) ; by convention, $\emptyset_r = \emptyset$. The Minkowski content of E is the quantity

$$\check{V}^+(E) := \liminf_{r \downarrow 0} \frac{\check{V}(E_r) - \check{V}(E)}{r} \in [0, \infty].$$

Lemma 7.1 Let E be a finite union of open intervals in (\check{B}, d) . Then

- (i) E is a Caccioppoli set in \check{B} ;
- (ii) $\check{V}^+(E) = P_{\check{g}}(E, \check{B})$.

Proof (i) The set E is a finite union of disjoint open intervals, \overline{E} is a finite union of closed intervals in \mathbb{R} with non-empty interior and separated by open sets in the sense of Theorem 6.1 and $\overline{E} = E \cup I$ for a finite set $I \subset \mathbb{R}$. So E is \mathcal{L}^1 -a.e. equivalent to $F := \overline{E} \cap \check{B}$; in particular, E is a Caccioppoli set in \check{B} by Theorem 6.1.

(ii) Suppose first that $F = [a_1, a_2] \subset \check{B}$. Then $E_r = F_r$ for each $r > 0$ and $\check{V}(E) = \check{V}(F)$ as \check{V} is non-atomic. For small $r > 0$,

$$\begin{aligned} \frac{1}{r} \check{V}(E_r \setminus E) &= \frac{1}{r} \check{V}(F_r \setminus F) = \frac{1}{r} \int_{a_1-r}^{a_1} \check{g} dy + \frac{1}{r} \int_{a_2}^{a_2+r} \check{g} dy \\ &\rightarrow \int_{\partial F} \check{g} d\mathcal{H}^0 = P_{\check{g}}(F, \check{B}) = P_{\check{g}}(E, \check{B}) \end{aligned}$$

as $r \downarrow 0$. The result for general E as in the statement follows from the property that the closed intervals in \overline{E} are separated by open sets. \square

For $p \in (0, V(B))$ we may write

$$\check{F}^{-1}(p) = \int_{V(B)/2}^p \frac{d\tau}{\check{J}(\tau)}$$

where $\check{J} = \check{g} \circ \check{F}^{-1}$. Note that $\check{J} = J$ due to the fact that $\check{F} \circ R = F$ on B .

Lemma 7.2 *Let E be an \mathcal{L}^1 -measurable set in \check{B} . Then $\check{V}(E_r) \geq \check{V}((E^*)_r)$ for each $r > 0$. In particular, the rearrangement \cdot^* is smoothing in the sense that $(E^*)_r \subset (E_r)^*$ for each \mathcal{L}^1 -measurable set E in \check{B} and $r > 0$.*

Proof We verify the conditions in [3, Theorem 2.1 (c)]. The measure \check{V} is a separable non-atomic Borel measure on the metric space (\check{B}, d) . The r -neighbourhood ($r > 0$) of any open ball in \check{B} is an open ball in \check{B} . Let E be a finite union of open intervals in (\check{B}, d) . By Lemma 7.1 and Corollary 6.11,

$$\check{V}^+(E) = P_{\check{g}}(E, \check{B}) \geq P_{\check{g}}(E^*, \check{B}) = \check{J}(\check{V}(E)).$$

Thus by [3, Theorem 2.1], $\check{V}(E_r) \geq \check{F}(\check{F}^{-1}(\check{V}(E)) + r) = \check{V}((E^*)_r)$ for any Borel set E in \check{B} with $0 < \check{V}(E) < V(B)$ and $r > 0$. The result then extends to \mathcal{L}^1 -measurable sets in \check{B} . \square

Lemma 7.3 *Let A, E be \mathcal{L}^1 -measurable sets in \check{B} with $A \subset E$. Then $d(A^*, \check{B} \setminus E^*) \geq d(A, \check{B} \setminus E)$.*

Here, $d(A, E) := \inf\{d(x, y) : x \in A, y \in E\}$ with the understanding that $\inf \emptyset = +\infty$.

Proof We use the criterion that for $r > 0$, $A_r \subset E$ if and only if $d(A, \check{B} \setminus E) \geq r$. Put $r := d(A, \check{B} \setminus E)$; we may assume that $r > 0$. By the criterion, $A_r \subset E$ and hence $(A_r)^* \subset E^*$. By Lemma 7.2, $\check{V}((A_r)^*) = \check{V}(A_r) \geq \check{V}((A^*)_r)$ meaning $(A^*)_r \subset (A_r)^* \subset E^*$ which entails that $d(A^*, \check{B} \setminus E^*) \geq r$ by the criterion. \square

The modulus of continuity of an arbitrary real-valued function u on \check{B} is defined by

$$\omega_u(t) := \sup \left\{ |u(x) - u(y)| : x, y \in \check{B} \text{ and } d(x, y) < t \right\} \in [0, \infty] \text{ for } t > 0.$$

Observe that u is uniformly continuous on \check{B} if and only if $\lim_{t \downarrow 0} \omega_u(t) = 0$. We state the following criterion without proof.

Lemma 7.4 *Let u be a real-valued function on \check{B} and $t, \tau > 0$. Then $\omega_u(t) > \tau$ if and only if there exist $s, s' \in \mathbb{R}$ with $s > s' + \tau$ such that $d(\{u > s\}, \check{B} \setminus \{u > s'\}) < t$.*

Proposition 7.5 *Assume (A.1)–(A.5).*

- (i) Let u be a real-valued \mathcal{L}^1 -measurable function on \check{B} . Then $\omega_u(t) \geq \omega_{u^*}(t)$ for each $t > 0$.
- (ii) If u is uniformly continuous on \check{B} then so is u^* .
- (iii) If u is Lipschitz continuous on \check{B} then so is u^* and $\text{Lip}(u^*, \check{B}) \leq \text{Lip}(u, \check{B})$.

Proof Let $t > 0$. We may assume that $\omega_{u^*}(t) > 0$. Choose $\tau > 0$ such that $\omega_{u^*}(t) > \tau$. By Lemma 7.4 there exist $s, s' \in \mathbb{R}$ with $s > s' + \tau$ such that $d(\{u^* > s\}, \check{B} \setminus \{u^* > s'\}) < t$. Now $\{u^* > s\} = \{|u| > s\}^*$ and likewise for s' by the counterpart of the equimeasurability property (6.4). By Lemma 7.3 we deduce that $d(\{|u| > s\}, \check{B} \setminus \{|u| > s'\}) < t$ and again by Lemma 7.4 that $\omega_u(t) \geq \omega_{|u|}(t) > \tau$. Item (i) then follows. Part (ii) is a ready consequence. As for (iii),

$$\begin{aligned} \text{Lip}(u, \check{B}) &:= \sup \left\{ \frac{|u(x) - u(y)|}{d(x, y)} : x, y \in \check{B}, x \neq y \right\} \\ &= \sup_{t>0} (1/t) \omega_u(t) \geq \sup_{t>0} (1/t) \omega_{u^*}(t) = \text{Lip}(u^*, \check{B}). \end{aligned}$$

□

Let u be a Lipschitz continuous function on (\check{B}, d) . By Rademacher's theorem (cf. [1, Theorem 2.14]) u is differentiable \mathcal{L}^1 -a.e. on \check{B} and its derivative coincides with the weak derivative on a set of full measure. Put

$$\begin{aligned} Z_1 &:= \{x \in \check{B} : u \text{ is differentiable at } x \text{ and } u'(x) = 0\}, \\ Z_2 &:= \{x \in \check{B} : u \text{ is not differentiable at } x\} \text{ and } Z := Z_1 \cup Z_2. \end{aligned}$$

By [1, Lemmas 2.95 and 2.96], $Z \cap \{u = t\} = \emptyset$ for \mathcal{L}^1 -a.e. $t \in \mathbb{R}$ and hence $N := u(Z) \subset \mathbb{R}$ is \mathcal{L}^1 -negligible. The analogous sets corresponding to u^* will be decorated with the subscript \star .

We shall make use of the coarea formula [1, Theorem 2.93 and (2.74)],

$$\int_{\check{B}} \phi |u'| dx = \int_{-\infty}^{\infty} \int_{\check{B} \cap \{u=t\}} \phi d\mathcal{H}^0 dt \quad (7.1)$$

for any \mathcal{L}^1 -measurable function $\phi : \check{B} \rightarrow [0, \infty]$.

Lemma 7.6 *Let u be a nonnegative Lipschitz continuous function on (\check{B}, d) . Then*

- (i) $\check{\mu}_u \in \text{BV}(\mathbb{R})$;
- (ii) $D\check{\mu}_u = -u_{\sharp} \check{V}$;
- (iii) $D\check{\mu}_u^a = D\check{\mu}_u \llcorner (\mathbb{R} \setminus N)$;
- (iv) $D\check{\mu}_u^s = D\check{\mu}_u \llcorner N$;
- (v) $A := \left\{ t \in \mathbb{R} : \mathcal{L}^1(Z \cap \{u = t\}) > 0 \right\}$ is the set of atoms of $D\check{\mu}_u$ and $D\check{\mu}_u^j = D\check{\mu}_u \llcorner A$;

(vi) $\check{\mu}_u$ is differentiable \mathcal{L}^1 -a.e. on \mathbb{R} with derivative given by

$$\check{\mu}'_u(t) = - \int_{(\check{B} \setminus Z) \cap \{u=t\}} \frac{\check{g}}{|u'|} d\mathcal{H}^0$$

for \mathcal{L}^1 -a.e. $t \in \mathbb{R}$;

(vii) $\text{Ran}(u) = \text{supp}(D\check{\mu}_u)$.

The notation above $D\check{\mu}_u^a$, $D\check{\mu}_u^s$, $D\check{\mu}_u^j$ stands for the absolutely continuous resp. singular resp. jump part of the measure $D\check{\mu}_u$ (see [1, 3.2] for example).

Proof For any $\varphi \in C_c^\infty(\mathbb{R})$,

$$\int_{-\infty}^{\infty} \check{\mu}_u \varphi' dt = \int_{\check{B}} \varphi \circ u d\check{V}$$

by Fubini's theorem; so $\check{\mu}_u \in \text{BV}(\mathbb{R})$ and $D\check{\mu}_u$ is the push-forward of \check{V} under u , $D\check{\mu}_u = -u_{\#} \check{V}$ (cf. [1, 1.70]). By (7.1),

$$D\check{\mu}_u \llcorner (\mathbb{R} \setminus N)(A) = -\check{V}(\{u \in A\} \cap \check{B} \setminus Z) = - \int_A \int_{(\check{B} \setminus Z) \cap \{u=t\}} \frac{\check{g}}{|u'|} d\mathcal{H}^0 dt$$

for any \mathcal{L}^1 -measurable set A in \mathbb{R} . In light of the above, we may identify $D\check{\mu}_u^a = D\check{\mu}_u \llcorner (\mathbb{R} \setminus N)$ and $D\check{\mu}_u^s = D\check{\mu}_u \llcorner N$. The set of atoms of $D\check{\mu}_u$ is defined by $A := \{t \in \mathbb{R} : D\check{\mu}_u(\{t\}) \neq 0\}$. By [13, Lemma 7.7], we may write A as in (v). The monotone function $\check{\mu}_u$ is a good representative within its equivalence class and is differentiable \mathcal{L}^1 -a.e. on \mathbb{R} with derivative given by the density of $D\check{\mu}_u$ with respect to \mathcal{L}^1 by [1, Theorem 3.28]. Item (vii) follows from (ii). \square

Lemma 7.7 *Let u be a nonnegative Lipschitz continuous function on (\check{B}, d) . Then $\int_{\check{B} \cap Z} u^2 d\check{V} = \int_{\check{B} \cap Z_{\star}} (u^*)^2 d\check{V}$.*

Proof As Z has finite \mathcal{L}^1 -measure, $A \subset \mathbb{R}$ is a countable set. Thus

$$\int_{\check{B} \cap Z} u^2 d\check{V} = \sum_{t \in A} t^2 \check{V}(Z \cap \{u = t\}) = \int_A t^2 d(u_{\#} \check{V}) = - \int_A t^2 dD\check{\mu}_u$$

and an analogous result holds for u^* by Lemma 7.6. The fact that $\check{\mu}_u = \check{\mu}_{u^*}$ entails that $A = A_{\star}$. This leads to the result. \square

Theorem 7.8 *Assume (A.1)–(A.5). Let u be a Lipschitz continuous function on (\check{B}, d) . Then $u, u^* \in D(\check{\mathcal{E}})$ and $\check{\mathcal{E}}(u, u) \geq \check{\mathcal{E}}(u^*, u^*)$.*

Proof Given a Lipschitz continuous function u on (\check{B}, d) , $u \in W^{1,\infty}(\check{B})$ and $\|u'\|_{L^\infty(\check{B})} = \text{Lip}(u, \check{B})$ (see [1, Proposition 2.13]) so $u \in D(\check{\mathcal{E}})$. The same is true for u^* by Proposition 7.5. Replacing u by $|u|$ and using the contraction property of the Dirichlet form $(D(\check{\mathcal{E}}), \check{\mathcal{E}})$ we may assume that u is nonnegative.

The proof hinges on the identity

$$\int_{\check{B} \setminus Z} (u^2 + |u'|^2) d\check{V} = \int_{\text{Ran}(u)} \int_{(\check{B} \setminus Z) \cap \{u=t\}} \frac{u^2 + |u'|^2}{|u'|} \check{g} d\mathcal{H}^0 dt, \quad (7.2)$$

which is a consequence of the coarea formula (7.1).

As $\check{\mu}_u = \check{\mu}_{u^*}$ we have that $D\check{\mu}_u = D\check{\mu}_{u^*}$. In particular, we derive that

$$\check{\mu}'_u(t) = - \int_{(\check{B} \setminus Z) \cap \{u=t\}} \frac{\check{g}}{|u'|} d\mathcal{H}^0 = - \int_{(\check{B} \setminus Z_*) \cap \{u^*=t\}} \frac{\check{g}}{|u^*'|} d\mathcal{H}^0 = \check{\mu}'_{u^*}(t) \quad (7.3)$$

for \mathcal{L}^1 -a.e. $t \in \mathbb{R}$ by Lemma 7.6.

Let $t \in \text{Ran}(u)$ be such that $Z \cap \{u = t\} = \emptyset$ and $\int_{(\check{B} \setminus Z) \cap \{u=t\}} \check{g} d\mathcal{H}^0 < \infty$ and the analogous properties hold for u^* . We assume in addition that (7.3) holds. Such t comprise a set of full measure in the range of u . Then $\partial\{u > t\} = \check{B} \cap \{u = t\} \cup \partial\check{B} \cap \partial\{u > t\}$ and $\{u > t\}$ is a Caccioppoli set in \check{B} with finite \check{g} -perimeter; and likewise for u^* . From Corollary 6.11,

$$\begin{aligned} \infty &> \int_{\check{B} \cap \{u=t\}} \check{g} d\mathcal{H}^0 = \int_{\partial\{u>t\}} \check{g} d\mathcal{H}^0 = P_{\check{g}}(\{u > t\}, \check{B}) \\ &\geq P_{\check{g}}(\{u^* > t\}, \check{B}) = \int_{\check{B} \cap \{u^*=t\}} \check{g} d\mathcal{H}^0. \end{aligned} \quad (7.4)$$

Moreover, $\check{B} \cap \{u^* = t\}$ consists of a singleton; thus,

$$\begin{aligned} \check{\mu}'_{u^*}(t) &= - \int_{(\check{B} \setminus Z_*) \cap \{u^*=t\}} \frac{\check{g}}{|(u^*)'|} d\mathcal{H}^0 \\ &= - \int_{\check{B} \cap \{u^*=t\}} \frac{\check{g}}{|(u^*)'|} d\mathcal{H}^0 = - \int_{\check{B} \cap \{u^*=t\}} \check{g} d\mathcal{H}^0 / \int_{\check{B} \cap \{u^*=t\}} |(u^*)'| d\mathcal{H}^0. \end{aligned} \quad (7.5)$$

By (7.3) and (7.5),

$$\begin{aligned} \int_{(\check{B} \setminus Z) \cap \{u=t\}} \frac{u^2 + |u'|^2}{|u'|} \check{g} d\mathcal{H}^0 &= \int_{(\check{B} \setminus Z) \cap \{u=t\}} \frac{u^2 + |u'|^2}{|u'|} \check{g} d\mathcal{H}^0 \\ &= \int_{(\check{B} \setminus Z) \cap \{u=t\}} \left\{ t^2 + |u'|^2 \right\} \frac{\check{g} d\mathcal{H}^0 / |u'|}{-\check{\mu}'_u(t)} (-\check{\mu}'_u(t)) \\ &\geq \left\{ t^2 + \left(\frac{\int_{(\check{B} \setminus Z) \cap \{u=t\}} \check{g} d\mathcal{H}^0}{-\check{\mu}'_u(t)} \right)^2 \right\} (-\check{\mu}'_u(t)) \\ &= \left\{ t^2 + \left(\frac{\int_{\check{B} \cap \{u=t\}} \check{g} d\mathcal{H}^0}{-\check{\mu}'_u(t)} \right)^2 \right\} (-\check{\mu}'_u(t)) \end{aligned}$$

$$\begin{aligned}
&\geq \left\{ t^2 + \left(\frac{\int_{\check{B} \cap \{u^* = t\}} \check{g} \, d\mathcal{H}^0}{-\check{\mu}'_{u^*}(t)} \right)^2 \right\} (-\check{\mu}'_{u^*}(t)) \\
&= \int_{\check{B} \cap \{u^* = t\}} \frac{(u^*)^2 + |(u^*)'|^2}{|(u^*)'|} \check{g} \, d\mathcal{H}^0
\end{aligned} \tag{7.6}$$

where Jensen's inequality has been used in the first inequality and (7.4) in the second. This inequality combined with (7.2) as well as Lemma 7.7 lead to the result. \square

Corollary 7.9 *Let $u \in D(\check{\mathcal{E}})$. Then $u^* \in D(\check{\mathcal{E}})$ and $\check{\mathcal{E}}(u, u) \geq \check{\mathcal{E}}(u^*, u^*)$.*

Proof Let $u \in D(\check{\mathcal{E}})$. By Lemma 5.5 we may choose a sequence (u_h) in $C^\infty(\overline{\check{B}})$ that converges to u in $(D(\check{\mathcal{E}}), \check{\mathcal{E}})$; each u_h is Lipschitz continuous on \check{B} . By Theorem 6.13, (u_h^*) converges to u^* in $L^2(\check{B}, \check{V})$. By Theorem 7.8, each $u_h^* \in D(\check{\mathcal{E}})$ and $\check{\mathcal{E}}(u_h^*, u_h^*) \leq \check{\mathcal{E}}(u_h, u_h)$, so the sequence $(\check{\mathcal{E}}(u_h^*, u_h^*))_h$ is uniformly bounded in \mathbb{R} . By the Banach-Alaoglu theorem (cf. [19, A2 Theorem 2.1]) we may assume that $u_h^* \rightarrow v$ weakly as $h \rightarrow \infty$ in $(D(\check{\mathcal{E}}), \check{\mathcal{E}})$ for some $v \in D(\check{\mathcal{E}})$ by selecting a subsequence if necessary. We may identify v with u^* thanks to the $L^2(\check{B}, \check{V})$ convergence and the Banach-Saks theorem (cf. [19, A2 Theorem 2.2]); hence $u^* \in D(\check{\mathcal{E}})$. By [16, Theorem 10.1.5],

$$\check{\mathcal{E}}(u^*, u^*) \leq \liminf_{h \rightarrow \infty} \check{\mathcal{E}}(u_h^*, u_h^*) \leq \liminf_{h \rightarrow \infty} \check{\mathcal{E}}(u_h, u_h) = \check{\mathcal{E}}(u, u).$$

\square

Corollary 7.10 *Let $u \in D(\hat{\mathcal{E}})$. Then $u^\star \in D(\hat{\mathcal{E}})$ and $\hat{\mathcal{E}}(u, u) \geq \hat{\mathcal{E}}(u^\star, u^\star)$.*

Proof Let $u \in D(\hat{\mathcal{E}})$. Then $v := u \circ R^{-1} \in D(\check{\mathcal{E}})$ by Lemma 5.5. Moreover, $u^\star = v^\star \circ R$ by Proposition 6.10. By Lemma 5.5 and Corollary 7.9,

$$\hat{\mathcal{E}}(u, u) = \check{\mathcal{E}}(v, v) \geq \check{\mathcal{E}}(v^\star, v^\star) = \hat{\mathcal{E}}(v^\star \circ R, v^\star \circ R) = \hat{\mathcal{E}}(u^\star, u^\star).$$

\square

8 Equality Case in the Pólya–Szegő Inequality

We now investigate the equality case in the Pólya–Szegő inequality.

Lemma 8.1 *Let $u \in D(\check{\mathcal{E}})$ and $t \in \mathbb{R}$.*

- (i) *Put $v := u \wedge t$. Then $\int_{\check{B}} |v'|^2 \, d\check{V} = \int_{\check{B} \cap \{u > t\}} |u'|^2 \, d\check{V} = \int_{\check{B} \cap \{u \geq t\}} |u'|^2 \, d\check{V}$.*
- (ii) *Put $v := u \vee t$. Then $\int_{\check{B}} |v'|^2 \, d\check{V} = \int_{\check{B} \cap \{u < t\}} |u'|^2 \, d\check{V} = \int_{\check{B} \cap \{u \leq t\}} |u'|^2 \, d\check{V}$.*

Proof We only prove (i). Write

$$\int_{\check{B}} |v'|^2 \, d\check{V} = \int_{\check{B} \cap \{u < t\}} |v'|^2 \, d\check{V} + \int_{\check{B} \cap \{u = t\}} |v'|^2 \, d\check{V} + \int_{\check{B} \cap \{u > t\}} |v'|^2 \, d\check{V}.$$

The set $\{u < t\}$ is open in \check{B} as u is continuous so $v = u$ and $v' = u'$ there. By [13, Lemma 7.7], $v' = 0$ \mathcal{L}^1 -a.e. on $\{u \geq t\}$. \square

Suppose that $u \in W_{\text{loc}}^{1,2}(\check{B})$ is precisely represented in the sense of [18, (2.5)]. Then the set $\{u = t\}$ is finite or countably infinite for \mathcal{L}^1 -a.e. $t \in \mathbb{R}$ and the coarea formula (7.1) holds for u by [18, Theorem 1.1]. With Z as before it follows that $Z \cap \{u = t\} = \emptyset$ for \mathcal{L}^1 -a.e. $t \in \mathbb{R}$ and hence $N := u(Z) \subset \mathbb{R}$ is \mathcal{L}^1 -negligible.

Lemma 8.2 *Let u be a nonnegative function in $W_{\text{loc}}^{1,2}(\check{B})$ precisely represented in the sense of [18, (2.5)]. Then statements (i)–(vii) of Lemma 7.6 hold.*

Proof This runs as in Lemma 7.6. \square

Lemma 8.3 *Let $u \in D(\check{\mathcal{E}})$ be nonnegative. Then $\int_{\check{B} \cap Z} u^2 d\check{V} = \int_{\check{B} \cap Z_\star} (u^\star)^2 d\check{V}$.*

Proof The proof proceeds as in Lemma 7.7. \square

Lemma 8.4 *Let $u \in D(\check{\mathcal{E}})$ be nonnegative. Then*

(i) *for $t', t'' \in \mathbb{R}$ with $0 \leq t' < t''$,*

$$\int_{\check{B} \cap \{t' < u \leq t''\}} (u^2 + |u'|^2) d\check{V} \geq \int_{\check{B} \cap \{t' < u \leq t''\}} ((u^\star)^2 + |(u^\star)'|^2) d\check{V};$$

(ii) *for \mathcal{L}^1 -a.e. $t \in \mathbb{R}$,*

$$\int_{(\check{B} \setminus Z) \cap \{\tilde{u} = t\}} \frac{\tilde{u}^2 + |\tilde{u}'|^2}{|\tilde{u}'|} \check{g} d\mathcal{H}^0 \geq \int_{(\check{B} \setminus Z_\star) \cap \{u^\star = t\}} \frac{(u^\star)^2 + |(u^\star)'|^2}{|(u^\star)'|} \check{g} d\mathcal{H}^0$$

where \tilde{u} is the unique continuous representative of u (cf. [1, Definition 3.31 and after]).

Proof (i) Put $v := (u \vee t') \wedge t'' \in D(\check{\mathcal{E}})$. By Lemma 8.1,

$$\int_{\check{B}} (v^2 + |v'|^2) d\check{V} = \int_{\check{B} \cap \{t' < u \leq t''\}} (u^2 + |u'|^2) d\check{V} + t'^2(1 - \mu_u(t')) + t''^2\mu_u(t'').$$

By Lemma 10.2, $\check{v}^\sharp = (\check{u}^\sharp \vee t') \wedge t''$ on $[0, \check{V}(\check{B})]$ and hence $v^\star = (u^\star \vee t') \wedge t''$. We may then write an identity of the above form but with u^\star, v^\star in place of u, v . The statement then follows from Corollary 7.9.

(ii) Note that \tilde{u} is precisely represented in the sense of [18, (2.5)]. By the coarea formula for Sobolev mappings [18, Theorem 1.1],

$$\int_{\check{B}} \phi |u'| dx = \int_{-\infty}^{\infty} \int_{\check{B} \cap \{\tilde{u} = t\}} \phi d\mathcal{H}^0 dt$$

for any \mathcal{L}^1 -measurable function $\phi : \check{B} \rightarrow [0, \infty]$. In particular,

$$\int_{\check{B} \setminus Z} \phi(u) (u^2 + |u'|^2) d\check{V} = \int_{-\infty}^{\infty} \int_{(\check{B} \setminus Z) \cap \{\tilde{u} = t\}} \left\{ \frac{\tilde{u}^2 + |\tilde{u}'|^2}{|\tilde{u}'|} \check{g} d\mathcal{H}^0 \right\} \phi(t) dt \quad (8.1)$$

for any \mathcal{L}^1 -measurable function $\phi : \mathbb{R} \rightarrow [0, \infty]$.

Define

$$w(t) := \int_{(\check{B} \setminus Z) \cap \{u > t\}} (u^2 + |u'|^2) d\check{V}$$

for $t \in \mathbb{R}$. For $\varphi \in C_c^\infty(\mathbb{R}, \mathbb{R})$,

$$\int_{-\infty}^{\infty} w\varphi' dx = \int_{\check{B} \setminus Z} \varphi(u)(u^2 + |u'|^2) d\check{V}$$

so that $w \in \text{BV}(\mathbb{R})$. By (8.1), $Dw = \rho \mathcal{L}^1$ where

$$\rho(t) := - \int_{(\check{B} \setminus Z) \cap \{\tilde{u}=t\}} \frac{\tilde{u}^2 + |\tilde{u}'|^2}{|\tilde{u}'|} \check{g} d\mathcal{H}^0$$

for $t \in \mathbb{R}$; that is, w is absolutely continuous [1, Definition 3.31]. By [1, Theorem 3.28], w is differentiable \mathcal{L}^1 -a.e. on \mathbb{R} and $w' = \rho$ \mathcal{L}^1 -a.e. on \mathbb{R} . The same holds for the function w_\star defined as for w but with u^\star in place of u . Note that $\tilde{u}^\star = u^\star$. The statement then follows from (i). \square

We state the following lemma without proof.

Lemma 8.5 *Let u be a continuous real-valued function on \check{B} . Suppose that for \mathcal{L}^1 -a.e. $t > 0$ the set $\{u > t\}$ is either an open interval in \check{B} abutting a boundary point or $\{u > t\} = \emptyset$. Then u is monotone on \check{B} .*

Theorem 8.6 *Let $u \in D(\check{\mathcal{E}})$ be nonnegative and suppose that $\check{\mathcal{E}}(u, u) = \check{\mathcal{E}}(u^\star, u^\star)$. Then \tilde{u} is monotone on \check{B} .*

Proof By (8.1),

$$\check{\mathcal{E}}(u, u) = \int_{\check{B} \cap Z} u^2 d\check{V} + \int_{\text{Ran}(\tilde{u})} \int_{(\check{B} \setminus Z) \cap \{\tilde{u}=t\}} \frac{\tilde{u}^2 + |\tilde{u}'|^2}{|\tilde{u}'|} \check{g} d\mathcal{H}^0 dt$$

and a similar identity holds for u^\star . We may assume that $\text{Ran}(\tilde{u})$ is a closed interval in $[0, \infty)$ with non-empty interior. By Lemmas 8.3 and 8.4,

$$\int_{(\check{B} \setminus Z) \cap \{\tilde{u}=t\}} \frac{\tilde{u}^2 + |\tilde{u}'|^2}{|\tilde{u}'|} \check{g} d\mathcal{H}^0 = \int_{(\check{B} \setminus Z_\star) \cap \{u^\star=t\}} \frac{(u^\star)^2 + |(u^\star)'|^2}{|(u^\star)'|} \check{g} d\mathcal{H}^0$$

for \mathcal{L}^1 -a.e. $t \in \text{Ran}(\tilde{u})$. The chain of inequalities in (7.6) is valid with \tilde{u} in place of u and we may replace the sign \geq with the equality sign. In particular,

$$\int_{\check{B} \cap \{\tilde{u}=t\}} \check{g} d\mathcal{H}^0 = \int_{\check{B} \cap \{u^\star=t\}} \check{g} d\mathcal{H}^0 \quad (8.2)$$

for \mathcal{L}^1 -a.e. $t \in \text{Ran}(\tilde{u})$. Suppose that $t \in \text{Ran}(\tilde{u})$ such that $Z \cap \{\tilde{u} = t\} = \emptyset$ and $\int_{\tilde{B} \cap \{\tilde{u}=t\}} \check{g} d\mathcal{H}^0 < \infty$. We remark that $\partial\{\tilde{u} > t\} = \check{B} \cap \{\tilde{u} = t\} \cup \partial\check{B} \cap \partial\{\tilde{u} > t\}$ and $\{\tilde{u} > t\}$ is a Caccioppoli set in \check{B} with finite \check{g} -perimeter $P_{\check{g}}(\{\tilde{u} > t\}, \check{B}) = \int_{\check{B} \cap \{\tilde{u}=t\}} \check{g} d\mathcal{H}^0 < \infty$. From the identity (8.2) we derive that $P_{\check{g}}(\{\tilde{u} > t\}, \check{B}) = P_{\check{g}}(\{u^* > t\}, \check{B})$ for \mathcal{L}^1 -a.e. $t \in \text{Ran}(\tilde{u})$. By Corollary 6.12 the set $\{\tilde{u} > t\}$ is either an open interval in \check{B} abutting a boundary point or $\{\tilde{u} > t\} = \emptyset$ for \mathcal{L}^1 -a.e. $t > 0$. The statement follows by Lemma 8.5. \square

Corollary 8.7 *Let $u \in D(\hat{\mathcal{E}})$ be nonnegative and suppose that $\hat{\mathcal{E}}(u, u) = \hat{\mathcal{E}}(u^\star, u^\star)$. Then \tilde{u} is monotone on B .*

Proof Put $v := u \circ R^{-1} \in D(\check{\mathcal{E}})$ by Lemma 5.5. Also, $u^\star = v^\star \circ R$ by Proposition 6.10. We have

$$\hat{\mathcal{E}}(u, u) = \check{\mathcal{E}}(v, v) \geq \check{\mathcal{E}}(v^\star, v^\star) = \hat{\mathcal{E}}(v^\star \circ R, v^\star \circ R) = \hat{\mathcal{E}}(u^\star, u^\star).$$

by Lemma 5.5 and Corollary 7.9. So $\check{\mathcal{E}}(v, v) = \check{\mathcal{E}}(v^\star, v^\star)$. By Theorem 8.6, \tilde{v} is monotone on \check{B} and hence \tilde{u} is monotone on B . \square

9 Application to Exchange Flow

Consider the positive definite bilinear form $(\mathcal{D}, \mathcal{E})$ in $L^2(B, \mathcal{L}^1)$ given by

$$\mathcal{E}(u, v) := \int_B u' v' dx \quad (u, v \in \mathcal{D} := \{u : u/\psi \in C^\infty(\overline{B})\}).$$

Here, $\psi := u_B$ is given by $\psi(x) = (1/2)(1 - |x|^2)$ for $x \in B$.

Lemma 9.1 (i) $(\mathcal{D}, \mathcal{E})$ is closable in $L^2(B, \mathcal{L}^1)$ with closure denoted $(D(\mathcal{E}), \mathcal{E})$;
 (ii) $(D(\mathcal{E}), \mathcal{E})$ is a symmetric Dirichlet form in $L^2(B, \mathcal{L}^1)$;
 (iii) $D(\mathcal{E}) = W_0^{1,2}(B)$.

Proof (i) Suppose (u_h) is a sequence in \mathcal{D} such that $u_h \rightarrow 0$ in $L^2(B, \mathcal{L}^1)$. We write $v \in \mathcal{D}$ in the form $v = \psi w$ for some $w \in C^\infty(\overline{B})$; so $v'' \in L^2(B, \mathcal{L}^1)$. An integration-by-parts gives

$$\int_B u_h' v' dx = \int_{\partial B} u_h v' \nu d\mathcal{H}^0 - \int_B u_h v'' dx = - \int_B u_h v'' dx \rightarrow 0$$

as $h \rightarrow \infty$ where $\nu = \pm 1$ is the unit exterior normal on ∂B . The statement follows by [19, Lemma I.3.4]. Then $(D(\mathcal{E}), \mathcal{E})$ is a symmetric closed form by definition (cf. [19, Definition I.2.3]). By [19, Proposition I.4.10 and II.2 (c)], $(D(\mathcal{E}), \mathcal{E})$ is a symmetric Dirichlet form and (ii) follows. Note that $\psi = u_B \in W_0^{1,2}(B)$. Thus $C_0^\infty(B) \subset \mathcal{D} \subset W_0^{1,2}(B)$. This proves (iii). \square

The transient Dirichlet space $(D(\mathcal{E}), \mathcal{E})$ has reference function ψ^{-1} and

$$\int_B |u| \psi^{-1} dx \leq \sqrt{\mathcal{E}(u, u)} \text{ for all } u \in D(\mathcal{E}). \quad (9.1)$$

Denote by $D(\mathcal{E})_e$ the extended Dirichlet space; that is, the family of \mathcal{L}^1 -measurable functions u on B such that $|u| < \infty$ \mathcal{L}^1 -a.e. and there exists an \mathcal{E} -Cauchy sequence (u_h) of functions in $D(\mathcal{E})$ such that $u_h \rightarrow u$ \mathcal{L}^1 -a.e. on B . By [12, Lemma 1.5.5], $(D(\mathcal{E})_e, \mathcal{E})$ is a Hilbert space. The identity (9.1) extends to $D(\mathcal{E})_e$ and $D(\mathcal{E})_e \subset L^1(B, \psi^{-1} \mathcal{L}^1)$.

In the notation of Sect. 5 we take $f = \psi$ and $g = \psi^{3/2}$. The conditions (A.1)–(A.5) are satisfied by Proposition 6.5. Let V be the measure $V := \psi \mathcal{L}^1$ on B . We work with the symmetric Dirichlet form

$$\hat{\mathcal{E}}(u, v) := \int_B (uv + \psi u' v') dV \quad (u, v \in D(\hat{\mathcal{E}}))$$

in $L^2(B, V)$ with domain

$$D(\hat{\mathcal{E}}) := \left\{ u \in L^2(B, V) : u \text{ is weakly differentiable on } B \text{ and } \psi^{1/2} u' \in L^2(B, V) \right\}.$$

Proposition 9.2 *The bijective mapping $\mathcal{D} \rightarrow C^\infty(\bar{B})$; $u \mapsto \bar{u} := u/\psi$ extends to a Hilbert space isomorphism $E : D(\mathcal{E})_e \rightarrow D(\hat{\mathcal{E}})$. In particular, for $u, v \in D(\mathcal{E})_e$ we have that*

$$\mathcal{E}(u, v) = \hat{\mathcal{E}}(\bar{u}, \bar{v}). \quad (9.2)$$

Proof Using integration-by-parts and the fact that $-\psi'' = 1$ on B gives

$$\begin{aligned} \int_B \bar{u}' \bar{v}' \psi^2 dx &= \int_B (u/\psi)' \bar{v}' \psi^2 dx \\ &= \int_B (\psi u' - u \psi') \bar{v}' dx \\ &= \int_B u' \bar{v}' \psi dx - \int_{\partial B} u \psi' \bar{v} \nu d\mathcal{H}^0 + \int_B (u \psi')' \bar{v} dx \\ &= \int_B u' \bar{v}' \psi dx + \int_B (u \psi')' \bar{v} dx \\ &= \int_B u' \bar{v}' \psi dx + \int_B (u' \psi') \bar{v} - u \bar{v} dx \\ &= \int_B u' v' dx - \int_B \bar{u} \bar{v} \psi dx \end{aligned}$$

where $\nu = \pm 1$ is the unit exterior normal on ∂B . This establishes (9.2) on \mathcal{D} . The map E extends to $D(\mathcal{E})$ by density of \mathcal{D} in $D(\mathcal{E})$ as does (9.2). Let $u \in D(\mathcal{E})_e$ and choose a sequence (u_h) in $D(\mathcal{E})$ such that $u_h \rightarrow u$ \mathcal{L}^1 -a.e. on B as $h \rightarrow \infty$ and (u_h) is a \mathcal{E} -Cauchy sequence. Then (\bar{u}_h) is a Cauchy sequence in $(D(\hat{\mathcal{E}}), \hat{\mathcal{E}})$ with limit

$v \in D(\hat{\mathcal{E}})$ say. Define $Eu := v$. This definition is well-defined and the identity (9.2) holds on $D(\mathcal{E})_e$. In particular, the mapping E is injective.

We show that E is a surjection. Let $v \in D(\hat{\mathcal{E}})$. We put $u := \psi v$ and claim that $u \in D(\mathcal{E})_e$. Then there exists a sequence (w_h) in $C^\infty(\bar{B})$ such that $v_h := w_h \circ R \rightarrow v$ in $(D(\hat{\mathcal{E}}), \hat{\mathcal{E}})$ as $h \rightarrow \infty$. Put $u_h := \psi v_h$ for each h . Now $u_h \in C^1(\bar{B})$ and $u_h = 0$ on ∂B for each h . This means that (u_h) is a sequence in $D(\mathcal{E})$. By (9.2), (u_h) is a \mathcal{E} -Cauchy sequence. As (v_h) converges to v in $L^2(B, V)$ we may assume that (v_h) converges to v V -a.e. on B by selecting a subsequence if necessary. Thus, (u_h) converges to u \mathcal{L}^1 -a.e. on B . This shows that $u \in D(\mathcal{E})_e$ and $Eu = v$. \square

Theorem 9.3 For any \mathcal{L}^1 -measurable set A in B , $J(A) \leq J(A^\star)$.

Proof Let A be an \mathcal{L}^1 -measurable set in B and put $u := u_A \in D(\mathcal{E})$ and $v := u_{A^\star} \in D(\mathcal{E})$. By Theorem 6.6, Proposition 9.2, the Cauchy–Schwarz inequality and Corollary 7.10,

$$\begin{aligned} J(A) &= \int_B \bar{u} \chi_A dV \leq \int_B \bar{u}^\star \chi_{A^\star} dV = \mathcal{E}(\psi \bar{u}^\star, v) = \hat{\mathcal{E}}(\bar{u}^\star, \bar{v}) \\ &\leq \hat{\mathcal{E}}(\bar{u}^\star, \bar{u}^\star)^{1/2} \hat{\mathcal{E}}(\bar{v}, \bar{v})^{1/2} \leq \hat{\mathcal{E}}(\bar{u}, \bar{u})^{1/2} \hat{\mathcal{E}}(\bar{v}, \bar{v})^{1/2} = J(A)^{1/2} J(A^\star)^{1/2} \end{aligned}$$

and the result follows. \square

Theorem 9.4 Suppose that A is an \mathcal{L}^1 -measurable set in B such that $J(A) = J(A^\star)$. Then A is \mathcal{L}^1 -a.e. equivalent to an open interval in B abutting a boundary point of B .

Proof We may assume that $0 < V(A) < V(B)$. From the chain of inequalities in the proof of Theorem 9.3 we derive that $\hat{\mathcal{E}}(\bar{u}^\star, \bar{u}^\star) = \hat{\mathcal{E}}(\bar{u}, \bar{u})$. Put $v := \bar{u}$. From Corollary 8.7 we infer that \tilde{v} is monotone on B . Put $p := V(A)/V(B) \in (0, 1)$. By Theorem 9.3, A is an optimal configuration for the problem (1.2) for the data (B, p) . By Proposition 3.3, $V(A \Delta \{\tilde{v} > c\}) = 0$ for some $c \in (0, 1)$. As \tilde{v} is monotone, $\{\tilde{v} > c\}$ is an open interval in B abutting an end-point of B . This leads to the result. \square

We may now characterise optimal configurations for the problem (1.2).

Theorem 9.5 Let $p \in (0, 1)$.

- (i) The sets $(-1, F^{-1}(pV(B)))$ and $(F^{-1}((1-p)V(B)), 1)$ are optimal configurations for the problem (1.2) with data (B, p) .
- (ii) If $E \subset B$ is an optimal configuration for the data (B, p) then E is \mathcal{L}^1 -a.e. equivalent to one of the sets in (i).

Let U be a bounded open connected set in \mathbb{R}^n ($n \geq 1$).

Proposition 9.6 (i) For $\lambda \in (-1, 1)$, $\gamma(U, \lambda) = 2I(U, p)$ where $p = (1 - \lambda)/2$.
(ii) $\gamma(U) = 2 \sup_{p \in (0, 1)} I(U, p)$.

Proof (i) Let A be an open set in U . Suppose u satisfies (1.3) and the condition $(u, 1) = 0$. Put $f := -(\lambda + 1)\chi_A - (\lambda - 1)\chi_{U \setminus A}$; then $u = Gf$. From the flux-balance condition and symmetry of the Green operator,

$$0 = (u, 1) = (\psi, f) = -(\lambda + 1)V(A) - (\lambda - 1)V(U \setminus A).$$

So $\lambda = V(U)^{-1}\{V(U \setminus A) - V(U)\}$ and $V(A) = \frac{1-\lambda}{2}V(U)$. We have

$$\begin{aligned} (\chi_{U \setminus A}, u) &= (G\chi_{U \setminus A}, f) \\ &= -(\lambda + 1)(G\chi_{U \setminus A}, \chi_A) - (\lambda - 1)(G\chi_{U \setminus A}, \chi_{U \setminus A}) \\ &= -(\lambda + 1)(V(A) - J(A)) - (\lambda - 1)(V(U \setminus A) - V(A) + J(A)) \\ &= 2J(A) - (\lambda + 1)V(A) - (\lambda - 1)V(U \setminus A) + (\lambda - 1)V(A) \\ &= 2J(A) + (\lambda - 1)V(A) = 2\left\{J(A) - V(U)p^2\right\} \end{aligned}$$

where $p = (1 - \lambda)/2$. This leads to the reformulation (i). (ii) follows immediately. \square

Recall that for $x, y \in B$,

$$G(x, y) = \begin{cases} (1/2)(1 - y)(1 + x) & \text{for } x \leq y; \\ (1/2)(1 + y)(1 - x) & \text{for } x \geq y. \end{cases}$$

Lemma 9.7 *The mapping $(0, 1) \rightarrow \mathbb{R}; p \rightarrow I(B, p)$ has a unique global maximum at $p = 1/2$.*

Proof Define $\eta : B \rightarrow \mathbb{R}$ by $\eta = J(A) - V(B)^{-1}V(A)^2$ where $A = (-1, t)$. Then $I(B, p) = \eta(F^{-1}(pV(B)))$ for $p \in (0, 1)$ by Theorem 9.5. For $t \in B$,

$$\begin{aligned} \frac{d}{dt}J((-1, t)) &= \frac{d}{dt} \int_{-1}^t G\chi_{(-1, t)}(x) dx = G\chi_{(-1, t)}(t) + \int_{-1}^t \frac{d}{dt}G\chi_{(-1, t)}(x) dx \\ &= G\chi_{(-1, t)}(t) + \int_{-1}^t G(x, t) dx = 2G\chi_{(-1, t)}(t), \end{aligned}$$

so that

$$\eta'(t) = 2G\chi_{(-1, t)}(t) - 2V(B)^{-1}V((-1, t))\psi(t).$$

A further computation gives

$$G\chi_{(-1, t)}(\xi) = (1/4)(1 - t)(1 + t)^2 \text{ and } V(B)^{-1}V((-1, t)) = (1/4)(2 + 3t - t^3).$$

We then obtain that $\eta'(t) = -(1/4)t(1 - t^2)^2$ for any $t \in B$. This proves the result. \square

Theorem 9.8 *Let $\lambda \in (-1, 1)$.*

- (i) *The sets $(-1, F^{-1}(pV(B)))$ and $(F^{-1}((1 - p)V(B)), 1)$ are optimal configurations for the problem (1.5) with data (U, λ) where $p = (1 - \lambda)/2$.*

- (ii) If $E \subset B$ is an optimal configuration for the problem (1.5) with data (U, λ) then E is \mathcal{L}^1 -a.e. equivalent to one of the sets in (i).
- (iii) The sets $(-1, 0)$ and $(0, 1)$ are optimal configurations for the problem (1.4).
- (iv) If $E \subset B$ is an optimal configuration for the problem (1.4) then E is \mathcal{L}^1 -a.e. equivalent to one of the sets in (iii).

Proof Parts (i) and (ii) follow from Proposition 9.6 and Theorem 9.5. (iii) and (iv) follow from Proposition 9.6 and Lemma 9.7. \square

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Appendix: On the Generalised Inverse Function

Let (X, \mathcal{A}, μ) be a finite measure space. Let $f : X \rightarrow [0, \infty)$ be an \mathcal{A} -measurable function. The distribution function $\mu_f : [0, \infty) \rightarrow [0, \mu(X)]$ of f is defined by $\mu_f(t) := \mu(\{f > t\})$ for $t \geq 0$. Note that μ_f is right-continuous and non-increasing on $[0, \infty)$ and $\mu_f(t) \rightarrow 0$ as $t \rightarrow \infty$. The generalised inverse $f^\sharp : [0, \mu(X)] \rightarrow [0, \infty]$ of μ_f is defined by $f^\sharp(s) := \inf\{t \geq 0 : \mu_f(t) \leq s\}$ with the understanding that $\inf \emptyset = +\infty$.

Lemma 10.1 *Let $f : X \rightarrow [0, \infty)$ be an \mathcal{A} -measurable function. Let $t \geq 0$ and $s \in [0, \mu(X)]$. Then*

- (i) $\mu_f(t) > s$ if and only if $f^\sharp(s) > t$;
- (ii) $\mu_f(t) \leq s$ if and only if $f^\sharp(s) \leq t$;
- (iii) $\mu_f(t-) < s$ then $f^\sharp(s) < t$ in case $t > 0$.

Proof Suppose $\mu_f(t) > s$. The set $\{f > s\} \subset [0, \infty)$ is relatively open by right-continuity of μ_f . It follows that $f^\sharp(s) > t$. Conversely, if $f^\sharp(s) > t$ then $\mu_f(t) > s$ by definition of f^\sharp . This shows (i). (ii) follows from (i). Suppose $t > 0$ and $\mu_f(t-) < s$. By definition of the left-handed limit, we may choose $0 < \tau < t$ such that $\mu_f(t-) \leq \mu_f(\tau) < s$. Then $f^\sharp(s) \leq \tau < t$. This shows (iii). \square

Lemma 10.2 *Let $f : X \rightarrow [0, \infty)$ be an \mathcal{A} -measurable function and $t > 0$. Then*

- (i) if $g := f \wedge t$ then $g^\sharp = f^\sharp \wedge t$ on $[0, \mu(X))$;
- (ii) if $g := f \vee t$ then $g^\sharp = f^\sharp \vee t$ on $[0, \mu(X))$.

Proof (i) We have that $\mu_g = \chi_{[0,t)}\mu_f$ on $[0, \infty)$ and $g^\sharp = t\chi_{[0,\mu_f(t-))} + f^\sharp\chi_{[\mu_f(t-),\mu(X))}$. On the other hand, $w := f^\sharp \wedge t = f^\sharp\chi_{\{f^\sharp < t\}} + t\chi_{\{f^\sharp \geq t\}}$. Let $0 \leq s < \mu_f(t-)$. As $\mu_f \geq \mu_f(t-) > s$ on $[0, t)$, $\{\mu_f \leq s\} \subset [t, \infty)$ and $f^\sharp(s) \geq t$; that is, $w(s) = t = g^\sharp(s)$. Now suppose that $s = \mu_f(t-)$. Note that $f^\sharp(s) \leq t$. Consider the case that $f^\sharp(s) = t$. Then $w(s) = t = f^\sharp(s) = g^\sharp(s)$. If $f^\sharp(s) < t$

then $w(s) = f^\sharp(s) = g^\sharp(s)$. Finally, suppose that $s > \mu_f(t-)$. By Lemma 10.1 (iii), $w(s) = f^\sharp(s) = g^\sharp(s)$. Item (i) follows.

(ii) In this case, $\mu_g = \chi_{[0,t)} + \chi_{[t,\infty)}\mu_f$ on $[0, \infty)$ and $g^\sharp = \chi_{[0,\mu_f(t))}f^\sharp + t\chi_{[\mu_f(t),\mu(X))}$. On the other hand, $w := f^\sharp \vee t = t\chi_{\{f^\sharp \leq t\}} + f^\sharp\chi_{\{f^\sharp > t\}}$. Let $0 \leq s < \mu_f(t)$. By Lemma 10.1 (i), $f^\sharp(s) > t$ and $w(s) = f^\sharp(s) = g^\sharp(s)$. Now suppose that $s \geq \mu_f(t)$. By Lemma 10.1 (ii), $f^\sharp(s) \leq t$ and $w(s) = t = g^\sharp(s)$. \square

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